p-ADIC DIFFERENTIAL EQUATIONS AND RAMIFICATION

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Let U be a connected open subset of the complex affine line $\mathbf{C}$ and let us consider a system of ordinary differential equations E with coefficients in the ring $\mathscr{O}(U)$ of regular functions on U. Let $z \in U$. By Cauchy's theorem, E admits a basis of solution in the neighborhood of z. Using the analytic continuation principle, it is easy to show that E even admits a basis of solutions on the biggest open disk centered in z that is contained in U. In other words, looking at the radii of convergence of the solutions around a point gives us the distance from this point to the boundary of U.

In the p-adic setting, the situation is totally different. If we consider the simple differential equation $y' = y$ on the affine line over $\mathbf{C}_p$ and the point $z = 0$, then the distance to the boundary is infinite, whereas the radius of convergence of a non-zero solution, a scalar multiple of the exponential function, is finite, equal to $|p|^{1/(p-1)}$. The goal of these notes is to explain that what may appear at first sight as a pathology actually gives rise to a rich invariant and that the radii of convergence may, for instance, be used to recover the ramification data associated to a morphism of curves.

Setting

Let $(k, |\cdot|)$ be a complete nonarchimedean valued field with non-trivial valuation. We will moreover assume that k is of characteristic 0 and algebraically closed.

Let X be a quasi-smooth k-analytic curve in the sense of Berkovich. Let $\mathscr{F}$ be a vector bundle of rank r on X endowed with a connection ∇. The k-vector space of horizontal sections will be denoted by $H^0(X, (\mathscr{F}, \nabla))$.

In more concrete terms, if we restrict to a closed disk D inside X, the data of the restriction of $(\mathscr{F}, \nabla)$ to D is equivalent to the data of a differential equation of the form $Y' = GY$ with $Y \in \mathscr{O}(D)^r$ and $G \in M_r(\mathscr{O}(D))$. Then, the space $H^0(D, (\mathscr{F}, \nabla))$ is the space of solutions of the differential equation.

1. Radii of convergence of differential equations

We want to define a family of radii of convergence associated to a point $x \in X$. We first consider the fundamental case where $x = 0$ and $X = D(0, 1^-)$.

We would like to define the radius of convergence at 0 as the radius of the biggest open disk centered at 0 on which the equation is trivial. This does not always exist and it is better to consider the supremum of the radii of the closed disks satisfying this property. Noting that requiring that the equation is trivial is equivalent to requiring that it admits

Date: July 10, 2017.

The author was partially supported by the ANR project "GLOBES": ANR-12-JS01-0007-01 and ERC Starting Grant "TOSSIBERG": 637027.

r linearly independant solutions, we may actually define a family of radii this way. The formal definition is as follows. For each $i \in \{1, \dots, r\}$, we set

$$R_i = \sup(\{R \in [0,1) \mid \dim_k(H^0(D(0,R^+),(\mathscr{F},\nabla))) \geq r-i+1\}).$$

We get a non-decreasing family of real numbers and the p-adic analogue of Cauchy's theorem ensures that R_1 is positive:

$$0 < R_1 \leq R_2 \leq \cdots \leq R_r \leq 1.$$

We can now extend the definition to an arbitrary point x in an arbitrary curve X. If x is k-rational, since the curve X is assumed to be quasi-smooth, it is locally isomorphic to an open disc in the neighborhood of x, hence we can reduce to the previous case.

For an arbitrary point $x \in X$, we can always find a complete valued extension K of k such that there exists a K-rational point x' in the curve X_K that lies over x. We can then define the radii of convergence at x to be the radii of convergence at x', hence reduce to the previous case.

When trying to make this definition precise, one runs into normalization issues, since there is no canonical open disk attached to a point. They can be dealt with thanks to a precise understanding of the fibers of a scalar extension.

Theorem 1 (J. Poineau - A. Pulita)**.** *Let K be a complete valued extension of the complete residue field $\mathscr{H}(x)$ of the point x. Denote by $\pi : X_K \to X$ the projection map. Then the Shilov boundary of the fiber $\pi^{-1}(x) \simeq \mathcal{M}(\mathscr{H}(x) \hat{\otimes}_k K)$ contains exactly one point, denoted by $\sigma_K(x)$, and the connected component of $\pi^{-1}(x) \setminus \{\sigma_K(x)\}$ are isomorphic to unit open disks in X_K.*

Moreover, if K is algebraically closed and maximally complete, then $Gal^c(K/k)$ acts transitively on the set of K-rational points of $\pi^{-1}(x)$.

Let $x \in X \setminus X(k)$ and choose a complete valued extension K of $\mathscr{H}(x)$. Let x' be a K-rational point in X_K over x and consider the connected component C of $\pi^{-1}(x)$ containing x'. One may then find a isomorphism between C and the open unit disk that sends x' to 0 and then use the definition given at the beginning of the section to define

$$0 < R_1(x) \leq \cdots \leq R_r(x) \leq 1.$$

The last part of the theorem ensures that it is independent of the choices.

The first definition of the radii of convergence in the p-adic case is due to B. Dwork in [Dwo73]. It predates Berkovich spaces and B. Dwork actually defined them for so-called "generic disks". In the setting of Berkovich spaces, the definition is due to F. Baldassarri, who used formal models to define suitable normalizations (see [Bal10]). A purely analytic definition (using triangulations as in [Duc14]) inspired by F. Baldassarri's has been worked out by J. Poineau and A. Pulita in [PP15]. The definition we use here is closer the latter: it can actually be recovered from it by truncation and renormalization using the radius of the point.

2. General results

A fundamental result about radii of convergence is the following.

Theorem 2 (J. Poineau - A. Pulita). *For each $i \in \{1, \dots, r\}$, the map*

$$R_i : X \setminus X(k) \to [0, 1]$$

is piecewise monomial and it has degree 1 in almost every direction outside a point $x \in X \setminus X(k)$.

Let us point out that using the true definition with triangulations, one can define the radii of convergence on the whole X and prove that they are continuous functions, see [PP15]

Let us now present some applications. The first one deals with the structure of modules with connections.

Theorem 3. *Let $x \in X \setminus X(k)$. If there exists $i \in \{1, \dots, r-1\}$ such that $R_i(x) < R_{i+1}(x)$, then, locally around x, the module with connection $(\mathscr{F}, \nabla)$ splits as a direct sum*

$$(\mathscr{F}, \nabla) = (\mathscr{F}', \nabla') \oplus (\mathscr{F}'', \nabla'),$$

where $(\mathscr{F}, \nabla)$ accounts for the first i radii and $(\mathscr{F}, \nabla)$ for the others.

The first versions of such a result have been obtained by P. Robba in [Rob75a] and B. Dwork and P. Robba in [DR77]. The version presented here is due to K. Kedlaya in [Ked15] and J. Poineau - A. Pulita in [PP13a].

As a second application, we want to explain how the radii of convergence enter the picture when we compute the de Rham cohomology of a module with connection (see [PP13b]).

Assume that X is a wide open curve and denote by $b_1, \dots, b_N$ its branches at infinity. Assume that the radii of convergence are monomial at infinity and, for each $j \in \{1, \dots, N\}$, define the irregularity of $(\mathscr{F}, \nabla)$ along b_j to be

$$\mathrm{Irr}_{b_j}(\mathscr{F}, \nabla) = -\sum_{i=1}^{r} \deg_{b_j}(R_i).$$

We assume moreover that $(\mathscr{F}, \nabla)$ is of geometric origin (the precise condition has to do with the exponents of the module and is automatically satisfied is there exists a Frobenius structure, for instance).

Theorem 4 (J. Poineau - A. Pulita). *In the previous situation, the de Rham cohomology groups $H^q(X, (\mathscr{F}, \nabla))$ are finite-dimensional and we have the following index formula:*

$$\chi_{dR}(X, (\mathscr{F}, \nabla)) = r\chi_c(X) - \sum_{j=1}^{N} \mathrm{Irr}_{b_j}(\mathscr{F}, \nabla),$$

where $\chi_c(X) = 2 - 2g(X) - N$ is the compactly supported Euler characteristic of X.

3. Pushforward of differential equations

Let Y be a quasi-smooth k-analytic curve. Let $\mathscr{E}$ be a vector bundle of rank r on Y endowed with a connection ∇. Let $\varphi : Y \to X$ be a finite étale morphism of degree d.

Let us first recall that ∇ induces a connection of $\varphi_*\mathscr{E}$ by pushforward. Indeed, the map

$$\mathscr{E} \xrightarrow{\nabla} \mathscr{E} \otimes \Omega^1_Y$$

induces a map

$$\varphi_*\mathscr{E} \xrightarrow{\varphi_*\nabla} \varphi_*(\mathscr{E} \otimes \Omega^1_Y) \simeq \varphi_*\mathscr{E} \otimes \Omega^1_X,$$

since $\Omega^1_Y \simeq \varphi^*\Omega^1_X$ because φ is étale.

This construction actually amounts to considering $(\mathscr{E}, \nabla)$ with the $\mathscr{O}_X$-module structure induced by φ.

We note that the rank of $\varphi_*\mathscr{E}$ is equal to dr and that, for each open subset U of X, we have

$$H^0(U, (\varphi_*\mathscr{E}, \varphi_*\nabla)) = H^0(\varphi^{-1}(U), (\mathscr{E}, \nabla)).$$

First example: tame case

Let $y \in Y \setminus Y(k)$ and set $x := \varphi(y)$. Assume that $\varphi^{-1}(x) = \{y\}$ and that the extension $\mathscr{H}(y)/\mathscr{H}(x)$ is tame. Then, we have

$$(R_1(x), \dots, R_{dr}(x)) = (\underbrace{R_1(y), \dots, R_1(y)}_{d \text{ times}}, \dots, \underbrace{R_r(y), \dots, R_r(y)}_{d \text{ times}}).$$

In order to prove the formula, let us extend the scalars to some big algebraically closed field K and consider a K-rational point x' as before. The extension $\mathscr{H}(\sigma_K(y))/\mathscr{H}(\sigma_K(x))$ is still tame, which implies that the morphism between the residue curve at $\sigma_K(y)$ (whose function field is $\widetilde{\mathscr{H}(\sigma_K(y))}$) and the residue curve at $\sigma_K(x)$ (whose function field is $\widetilde{\mathscr{H}(\sigma_K(x))}$) is separable, hence generically étale. In particular, since $\tilde{K}$ is algebraically closed, it is of degree 1 on almost every $\tilde{K}$-rational point. Using the fact that the $\tilde{K}$-rational points of the residue curve at $\sigma_K(x)$ correspond to the branches outside x, and the ramification index at a point to the degree on the corresponding branch, we deduce that the preimage by φ_K of the connected component C of $\pi^{-1}(x) \setminus \{\sigma_K(x)\}$ containing x' is a disjoint union of d connected components of $\pi^{-1}(y) \setminus \{\sigma_K(y)\}$, all of them isomorphic to C *via* φ_K. In particular, the pushforward module with connection on C is isomorphic to a direct sum of d summands, all isomorphic to the module with connection upstairs, and the result follows.

Second example: the Frobenius map

Assume that the residue characteristic is positive, $\mathrm{char}(\tilde{k}) = p > 0$, and that φ is the p^{th}-power map $\varphi : x \in \mathbf{A}^{1,\mathrm{an}}_k \mapsto x^p \in \mathbf{A}^{1,\mathrm{an}}_k$. Denote by y and x the Gauß points upstairs and downstairs respectively. Note that we have $\varphi^{-1}(x) = \{y\}$.

To proceed, we will need some information about the topological and metric behavior of the morphism φ around y. Let K be a complete valued extension of $\mathscr{H}(x)$ that is algebraically closed. Let x' be a K-rational point over x. For, $t \in (0, 1)$, denote by $\eta_{x',t}$ the unique point of the Shilov boundary of the closed disk with center x' and radius t. For convenience, set $\eta_{x',0} = x'$ and $\eta_{x',1} = \sigma_K(x)$. Let y' be a K-rational point whose image by φ_K is x' and define similarly $\eta_{y',t}$ for $t \in [0,1]$. Let $f : [0,1] \to [0,1]$ be the map such that, for each $t \in [0,1]$, we have $\eta_{x',f(t)} = \varphi_K(\eta_{y',t})$.

Remark 5. The function f defined above is an example of a profile function, according to M. Temkin's terminology (see [Tem14]). It is straightforward to extend the definition to any finite morphism of curves $\varphi : Y \to X$ and any point $y \in Y \setminus Y(k)$. Note that this is not M. Temkin's original definition, which expresses it using the restriction of φ along a "generic" path from a k-rational point $x' \in X(k)$ to x.

In this case, the topological behavior of φ is well-known and the profile function f can be computed explicitly:

$$\forall t \in [0,1],\ f(t) = \begin{cases} |p|t \text{ if } t < |p|^{1/(p-1)}; \\ t^p \text{ if } t \geq |p|^{1/(p-1)}. \end{cases}$$

• *Computation for* $(\mathscr{E}, \nabla) = (\mathscr{O}, d)$

In this case, the solutions are the constant functions, hence the maximal number of linearly independent solutions is exactly the number of connected components of the space under consideration. Based on the previous computations, we deduce that

$$\begin{aligned} \dim_K H^0(D(x',t^-), ((\varphi_K)_*\mathscr{O}, (\varphi_K)_*d)) &= \dim_K H^0(\varphi_K^{-1}(D(x',t^-)), (\mathscr{O}, d)) \\ &= \begin{cases} p \text{ if } t < |p|^{p/(p-1)}; \\ 1 \text{ if } t \geq |p|^{p/(p-1)}. \end{cases} \end{aligned}$$

It follows that we have

$$R_1(x) = \cdots = R_{p-1}(x) = |p|^{p/(p-1)} < R_p(x) = 1.$$

• *Computation for* $(\mathscr{E}, \nabla)$ *of rank one*

We now assume that $(\mathscr{E}, \nabla)$ has rank one, so that there is exactly one radius of convergence at y. The radii of convergence can then be computed as before, keeping in mind that $(\mathscr{E}, \nabla)$ is trivial on each closed disk with center y and radius $t < R_1(y)$.

Assume that $R_1(y) > |p|^{1/(p-1)}$. Then, we have

$$R_1(x) = \cdots = R_{p-1}(x) = |p|^{p/(p-1)} < R_p(x) = R_1(y)^p.$$

Assume that $R_1(y) \leq |p|^{1/(p-1)}$. Then, we have

$$R_1(x) = \cdots = R_{p-1}(x) = R_p(x) = |p|\, R_1(y).$$

Those results have first been obtained by K. Kedlaya in a purely algebraic manner, using the fact that the radii of convergence may be computed as the spectral radii of some operators (see [Ked10]). The geometric approach given here has the advantage that it is easily extendable to more general situations, provided that one understands the topological/metric behavior of the morphism.

Theorem 6 (V. Bojković - J. Poineau)**.** *Let* $x \in X \setminus X(k)$. *Set* $\varphi^{-1}(x) = \{y_1, \ldots, y_m\}$. *There exists an explicit formula computing the radii of convergence of* $(\varphi_*\mathscr{E}, \varphi_*\nabla)$ *at* x *in terms of*

i) the radii of convergence of $(\varphi_*\mathscr{E}, \varphi_*\nabla)$ *at the* y_i'*s;*
ii) the profile functions and the local degrees of φ *at the* y_i*'s.*

The aforementioned formula being quite intricate and requiring many notation, we omit it here and refer to the original paper [BP17] instead.

Let us quote a very important interpretation of the profile function from [Tem14]. Let $y \in Y \setminus Y(k)$ and assume that the extension $\mathscr{H}(y)/\mathscr{H}(x)$, where $x := \varphi(y)$ is Galois. Then, the profile function of φ at y coincides with the Herbrand function of the extension $\mathscr{H}(y)/\mathscr{H}(x)$ (in a multiplicative and suitably normalized version).

Recall that the Herbrand function is the one that is used to pass from the lower ramification filtration to the upper one. The same reasoning as above shows that, for an arbitrary étale morphism φ, the radii of convergence of $(\varphi_*\mathscr{O}, \varphi_* d)$ at a point $x \in X \setminus X(k)$ are the images of the break-points of the profile functions at the preimages of x. Combining those two facts, we deduce the following result.

Corollary 7. *Let* $y \in Y \setminus Y(k)$*. Set* $x := \varphi(y)$ *and assume that the extension* $\mathscr{H}(y)/\mathscr{H}(x)$ *is Galois. Then, the radii of convergence of* $(\varphi_*\mathscr{O}, \varphi_* d)$ *at* x *are precisely the upper jumps of the ramification filtration of the extension* $\mathscr{H}(y)/\mathscr{H}(x)$*.*

Last, we would like to mention a nice theoretical way to reformulate Theorem 6. To this end, let us introduce the profile of a module with connection $(\mathscr{E}, \nabla)$ on a curve Y. For $y \in Y \setminus Y(k)$, we call profile function of $(\mathscr{E}, \nabla)$ at y the continuous piecewise monomial function $f^y_{(\mathscr{E},\nabla)} : [0,1] \to [0,1]$ satisfying $f^y_{(\mathscr{E},\nabla)}(0) = 0$, $f^y_{(\mathscr{E},\nabla)}(1) = 1$ and

$$\forall s \in (0,1],\ \deg^-_{f^y_{(\mathscr{E},\nabla)}}(s) = \dim_k H^0\big(D(y, s^-), (\mathscr{E}, \nabla)\big).$$

It is not difficult to see that the function $f^y_{(\mathscr{E},\nabla)}$ and the family of radii of convergence $(\mathcal{R}_i(y, (\mathscr{E}, \nabla)))_{1\le i\le r}$ can be recovered from each other.

For $y \in Y \setminus Y(k)$, let us denote by f^y_φ and $n_\varphi(y)$ the profile function (see Remark 5) and the local degree of φ at y respectively. First, note that the profile function of φ is closely related to the profile function of the pushforward of the trivial module with connection.

Lemma 8. *Let* $y \in Y \setminus Y(k)$ *such that* $\varphi^{-1}(\varphi(y)) = \{y\}$*. Then, we have*

$$f^{\varphi(y)}_{\varphi_*(\mathscr{O},d)} = \big((f^y_\varphi)^{-1}\big)^{n_\varphi(y)}.$$

Let us now give rewrite, and make precise, Theorem 6 in this setting.

Theorem 9 (V. Bojković - J. Poineau)**.** *Let* $x \in X \setminus X(k)$*. Then, we have*

$$f^x_{(\varphi_*\mathscr{E},\varphi_*\nabla)} = \prod_{y\in\varphi^{-1}(x)} \big(f^y_{(\mathscr{E},\nabla)} \circ (f^y_\varphi)^{-1}\big)^{n_\varphi(y)}.$$

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