

# PUSHFORWARDS OF $p$ -ADIC DIFFERENTIAL EQUATIONS

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ABSTRACT. Given a differential equation on a smooth  $p$ -adic analytic curve, one may construct a new one by pushing forward by an étale morphism. The main result of the paper provides an explicit formula that relates the radii of convergence of the solutions of the two differential equations using invariants coming from the topological behavior of the morphism. We recover as particular cases the known formulas for Frobenius morphisms and tame morphisms.

As an application, we show that the radii of convergence of the pushforward of the trivial differential equation at a point coincide with the upper ramification jumps of the extension of the residue field of the point given by the morphism. We also derive a general formula computing the Laplacian of the height of the Newton polygon of a  $p$ -adic differential equation.

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## 1. INTRODUCTION

The theory of  $p$ -adic differential equations has seen quite a progress in the last years, not only in better understanding of the classical situation — equations with coefficients analytic functions on the affine line — but also in extending many classical results to a more general setting, where coefficients are now analytic functions on curves of higher genus (see [Bal10, Pul15, PP15, Ked15]). Berkovich theory of analytic spaces provided a fertile ground to do so, as, in this setting, the generic point, which is one of the most important notion in the classical theory, becomes a true point of the underlying space.

In order to extend classical results to  $p$ -adic differential equations on higher genus curves, one of the dominating ideas is to pushforward the situation into the affine line by a suitable map so that the change of invariants (such as the radii of convergence of solutions) can be tracked explicitly. To be more precise, let us fix an algebraically closed complete nonarchimedean nontrivially valued field  $k$  of characteristic 0. If we are given a quasi-smooth Berkovich curve  $Y$ , a differential equation  $(\mathcal{E}, \nabla)$  on  $Y$  and a point  $y \in Y$ , in order to understand the behavior of  $(\mathcal{E}, \nabla)$  around  $y$ , we look for a finite étale map  $\varphi : Y \rightarrow X$ , where now  $X$  is a domain in  $\mathbb{A}_k^1$  and study the equation  $\varphi_*(\mathcal{E}, \nabla)$  around the point  $x = \varphi(y)$ . If we can find a map  $\varphi$  such that we can precisely say how the invariants of  $(\mathcal{E}, \nabla)$  change when passing to  $\varphi_*(\mathcal{E}, \nabla)$ , we may substitute our original problem with, presumably, an easier one.

A celebrated example of such a map  $\varphi$  is the Frobenius endomorphism of the affine line  $\mathbb{A}_k^1$ , when  $k$  has positive residue characteristic  $p$ , that acts on the elements of  $k$  by simply raising to the power  $p$ . In this case, one can describe precisely how the radii of convergence get modified (*cf.* [Ked10, Theorem 10.5.1]), and this has been vastly exploited in bridging two relatively well understood classes of  $p$ -adic differential equations: the one where the radius of convergence at a given point is equal to 1, and the one where all the local solutions have a small radius of convergence (smaller than  $p^{1/(p-1)}$ ). Unfortunately, Frobenius pushforward is only applicable for differential equations on domains of  $\mathbb{A}_k^1$  (although an extension to higher genus curves has been announced in [BK]).

Another example, quite orthogonal to the former, where we can explicitly say how the radii of convergence of solutions change when passing from  $y$  to  $x = \varphi(y)$  is the one where the degree of the extension  $[\mathcal{H}(y) : \mathcal{H}(x)]$  is prime to the residue characteristic of  $k$ . It has been extensively used in [PP15] to establish continuity properties of radii of convergence of solutions of differential equations on curves of higher genus by pushing forward to the line, where such properties had previously been established in [Pul15]. A similar example has been used in [PP13] to prove harmonicity properties of the radii of convergence, subject to the quite unnatural assumption that the residue characteristic is different from 2 (in order to be able to impose the condition on the degree of the extensions in all directions out of a given point).

Of course, there is a separate intrinsic interest in describing the invariants of the pushforward of a differential equation by a finite étale map  $\varphi$  as above (for  $X$  not necessarily in  $\mathbb{A}_k^1$ ) in terms of the original equation, especially in describing the change of irregularity. To the best of our knowledge, very little is known so far.

Another aspect is the relation between  $p$ -adic differential equations and ramification of morphisms. It is long believed that there is a quite close relation between the pushforward of the *constant* differential equation by a morphism  $\varphi$  (that is  $\varphi_*(\mathcal{O}_Y, d_Y)$ ) and ramification properties of  $\varphi$ . Perhaps a first paper exploring this relation is [Bal13], where the author studied the bound on the radius of convergence in terms of the *different* of the morphism at a point, consequently deducing a variant of the  $p$ -adic Rolle theorem. The recent preprint [Tem14] by Michael Temkin introduced a refined invariant of a morphism at a non-rational point  $y \in Y$ , called a profile function (roughly speaking, it is a map of the real unit interval to itself that describes the morphism locally around  $y$ ). Furthermore, he developed a ramification theory for Galois extensions of one-dimensional analytic fields, introducing Herbrand function for such extensions. One of the main results in *ibid.* is that the Herbrand function of the extension  $\mathcal{H}(y)/\mathcal{H}(\varphi(y))$  coincides with the profile function of  $\varphi$  at  $y$ .

This is where the goal of the present article fits in: given a finite étale morphism  $\varphi : Y \rightarrow X$  of quasi-smooth  $k$ -analytic curves, points  $y \in Y$  and  $x = \varphi(y) \in X$  and a differential equation  $(\mathcal{E}, \nabla)$  on  $Y$ , describe the radii of convergence of solutions of  $\varphi_*(\mathcal{E}, \nabla)$  at  $x$  in terms of the radii of convergence of solutions of  $(\mathcal{E}, \nabla)$  at  $y$  and the profile function of the morphism  $\varphi$  at  $y$ . The goal is achieved in Theorem 4.3 where  $y$  is taken to be a rational point and  $Y$  and  $X$  are open unit discs, and in Theorem 4.6 for general  $Y$ ,  $X$  and  $y \in Y$ . Along the way, we provide simplified formulas for some special cases of  $(\mathcal{E}, \nabla)$  such as the constant connection. All the known instances of our results, such as the ones described above, are summarized in Corollary 4.8 and we hope to convince the reader that they can be easily recovered using our arguments.

We restrict ourselves to applications in two (apparently) different directions, namely ramification of one-dimensional analytic fields and irregularities and Laplacians in the area of  $p$ -adic differential equations. In the former case, we establish a close (expected) relation between the Herbrand function of a Galois extension of one-dimensional analytic fields  $\mathcal{H}(y)/\mathcal{H}(x)$  (for a non-rational point  $y \in Y$ ) and the radii of convergence at  $x$  of local sections at of the pushforward of the constant connection  $\varphi_*(\mathcal{E}, \nabla)$  (Corollary 5.3). In the latter case, we provide a formula for the change of irregularity and Laplacians when passing from  $(\mathcal{E}, \nabla)$  to  $\varphi_*(\mathcal{E}, \nabla)$ , and use it prove harmonicity properties of convergence polygons on arbitrary curves (Corollary 5.11).

The paper is structured as follows. In the second section we recall the properties of the profile of a morphism  $\varphi$  at a (non-rational) point  $y \in Y$ . Along the way, we provide

an alternative definition of the profile function based on the structure of the fibers of projection maps coming from extensions of scalars.

The third section is reserved for recalling the main definitions and results concerning  $p$ -adic differential equations on Berkovich curves.

The fourth and fifth sections are the core of the paper. In the fourth section, we provide the main results about the radii of pushforward of differential equations, while the fifth contains applications of our main formulas, as described above.

## 2. FINITE MORPHISMS BETWEEN QUASI-SMOOTH $k$ -ANALYTIC CURVES

Let  $k$  be an algebraically closed field, complete with respect to a non-archimedean and non-trivial valuation and of characteristic 0. The corresponding norm on the field is denoted by  $|\cdot|$  and the residue field is denoted by  $\tilde{k}$ .

All the curves throughout the paper will be  $k$ -analytic in the sense of Berkovich. In addition, we will assume that they are quasi-smooth (rig-smooth) and unless otherwise stated we will assume that they are connected too.

**2.1. Quasi-smooth Berkovich curves.** We will use for granted the construction of the  $k$ -analytic affine line  $\mathbb{A}_k^1$  from [Ber90, Section 4.2].

For  $a \in k$  and  $r \in [0, \infty)$ , we denote by  $D(a, r)$  (resp.  $D(a, r^-)$  if  $r > 0$ ) the Berkovich closed (resp. open) disc centered at point  $a$  and of radius  $r$ . Similarly, for  $a \in k$  and  $r_1 \geq r_2 \in (0, \infty)$  (resp.  $r_1 > r_2 \in (0, \infty)$ ) we denote by  $A[a; r_2, r_1]$  (resp.  $A(a; r_2, r_1)$ ) the Berkovich closed (resp. open) annulus centered in the point  $a$  and of inner radius  $r_2$  and outer radius  $r_1$ . The Shilov point of the disc  $D(a, r)$  will be denoted by  $\eta_{a,r}$  or  $\eta_r$  if  $a = 0$ .

Note that the disc  $D(a, r)$  is strictly  $k$ -affinoid (we will simply say strict) if and only if  $r \in |k^*|$ . By abuse of notation we will say that an open disc  $D(a, r^-)$  is strict if  $r \in |k^*|$ . In this case it is isomorphic to the open unit disc  $D(0, 1^-)$ . Similarly, we say that the open annulus  $A(a; r_2, r_1)$  is strict if  $r_1, r_2 \in |k^*|$ ; in this case, it is isomorphic to the normalized annulus  $A(0; r, 1)$ , where  $r = r_2/r_1 \in |k^*|$ .

In general, given a  $k$ -analytic curve  $X$ , an analytic domain of  $X$  isomorphic to an (strict) open disc (resp. (strict) open annulus) is called an (strict) open disc (resp. (strict) open annulus). If  $D$  (resp.  $A$ ) is a strict open disc (resp. strict open annulus), then any isomorphism  $T : D \xrightarrow{\sim} D(0, 1)$  (resp.  $T : A \xrightarrow{\sim} A(0; r, 1)$ ) will be called a normalizing coordinate on  $D$  (resp. a normalizing coordinate on  $A$ ).

The skeleton  $S(A)$  of an annulus  $A$  is the complement of all the open discs in  $A$ . For example, if  $A = A(a; r_2, r_1)$ , then  $S(A) = \{\eta_{a,r} \mid r \in (r_2, r_1)\}$  and it can be naturally identified with the interval  $(r, 1)$ , where  $r = r_2/r_1$ . This interval only depends on  $A$  and not on the choices of  $r_2$  and  $r_1$ . In particular,  $S(A)$  is naturally endowed with a metric.

It follows from their explicit descriptions that discs and annuli are endowed with (real) tree structures and that each interval in them containing no type 1 point naturally carries a metric modeled on that on the skeleton of an annulus above.

More generally, it follows from the semistable reduction theorem that, up to removing a locally finite set, any quasi-smooth  $k$ -analytic curve is a disjoint union of discs and annuli. As a consequence, quasi-smooth  $k$ -analytic curves are endowed with (real) graphs structures. In particular, it makes sense to speak of intervals in them and, as before, any interval containing no type 1 point naturally carries a metric.

Now, let  $x \in X$  be a point. The set of germs of intervals in  $X$  with an endpoint in  $x$  is denoted by  $T_x X$  and is called the set of tangent directions (or tangent space) at  $x$ . An element of the  $T_x X$  is also called a branch.

Let  $K/k$  be a complete valued extension of  $k$  and let us put  $X_K := X \widehat{\otimes}_k K$  (see [Ber90, 2.1] for the construction of  $X \widehat{\otimes}_k K$ ). Then, we have a natural projection map  $\pi_{K/k} : X_K \rightarrow X$  (sometimes denoted by  $\pi_K$  if  $k$  is understood from the context). If  $x$  is a point in  $X$ , then every  $K$ -rational point in the set  $\pi_{K/k}^{-1}(x)$  will be called a  $K$ -rational point above  $x$  and we use notation  $x_K$  for them. For more details one may refer to [PP15, Section 2.2] or [BK, Section 3.1].

The previous construction and notion are meaningful even if the base field  $k$  is not algebraically closed but rather just a complete valued field. However, by [Poi13, Corollaire 3.14], when  $k$  is algebraically closed, each point  $x \in X$  is *peaked* (see [Ber90, Section 5.2]) or *universal* ([Poi13, Définition 3.2]): for each complete valued extension  $K/\mathcal{H}(x)$ , where  $\mathcal{H}(x)$  is the complete residue field of  $x$ , the tensor norm on  $\mathcal{H}(x) \widehat{\otimes}_k K$  is multiplicative. In particular,  $\mathcal{M}(\mathcal{H}(x) \widehat{\otimes}_k K)$  contains a distinguished point, denoted by  $\sigma_{K/k}(x)$  (or simply  $\sigma_K(x)$  if  $k$  is clear from the context), which corresponds to the (multiplicative) norm on  $\mathcal{H}(x) \widehat{\otimes}_k K$ . By [Poi13, Corollaire 3.7], the map of topological spaces  $\sigma_{K/k} : X \rightarrow X_K$  is a continuous retraction of  $\pi_{K/k}$ .

**Definition 2.1.** *We call the point  $\sigma_{K/k}(x)$  the  $K$ -generic point above  $x$  in  $X_K$ .*

**2.2. Radial morphisms and profile functions.** We recall the notions of radial morphisms and profile functions introduced in [Tem14] and to which we refer for more details.

**2.2.1. Radial morphisms of open discs.** First we set up some notation that will be used later on.

Consider the Berkovich affine line  $\mathbb{A}_k^1$  and choose a coordinate  $T$  on it. Denote by  $r$  the radius function: for each  $x \in \mathbb{A}_k^1$ , we have

$$r(x) := \inf_{a \in k} |(T - a)(x)|.$$

Let  $D$  be a strict open disc and let  $z \in D(k)$ . Choose an isomorphism  $T_z : D \xrightarrow{\sim} D(0, 1^-)$  between  $D$  and the open unit disc in  $\mathbb{A}^1$  that sends  $z$  to 0.

For each  $\rho \in (0, 1]$ , set  $D(z, \rho^-) = T_z^{-1}(D(0, \rho^-))$ . Similarly, for each  $\rho \in (0, 1)$ , set  $D(z, \rho) = T_z^{-1}(D(0, \rho))$  and denote by  $z_\rho$  the unique point of the Shilov boundary of  $D(z, \rho)$ . We also set  $D(z, 0) = \{z\}$  and  $z_0 = z$ . We denote by  $l_z$  the canonical path

from  $z$  to the “boundary” of  $D$ , that is

$$l_z := \{z_\rho, \rho \in [0, 1)\}.$$

All these definitions are independent of the choice of  $T_z$ .

We define a radius function  $r$  on  $D$  by pulling back the restriction of the radius function to  $D(0, 1^-)$ . It is independent of the choice of  $z$  and  $T_z$ .

If  $\varphi : D_2 \rightarrow D_1$  is a finite morphism of strict open unit, then, for each point  $z \in D_2(k)$ ,  $\varphi$  induces a continuous increasing map  $f_{\varphi, z} : [0, 1) \rightarrow [0, 1)$  given by  $\rho \mapsto r(\varphi(z_\rho))$ . We sometimes extend this map to a continuous map  $[0, 1] \rightarrow [0, 1]$  by setting  $f_{\varphi, z}(1) = 1$ . We denote it the same way.

A closely related function is given in the next definition.

**Definition 2.2.** *Let  $\varphi : Y \rightarrow X$  be a finite morphism of  $k$ -analytic curves. Let  $y \in Y$ . We define the multiplicity of  $\varphi$  at  $y$  as*

$$(1) \quad n_y = e \cdot [\mathcal{H}(y) : \mathcal{H}(\varphi(y))],$$

where  $e = 1$  if  $y$  has type 2, 3 or 4 and  $e$  is the unique integer such that  $m_y = m_{\varphi(y)}^e \mathcal{O}_y$  if  $y$  has type 1. We define the multiplicity function of  $\varphi$  by

$$n_\varphi : y \in Y \mapsto n_y \in \mathbb{N}.$$

Let  $I$  be an interval in  $\mathbb{R}_+$  whose interior  $\overset{\circ}{I}$  is not empty. A map  $f : I \rightarrow \mathbb{R}_+$  is called  $|k^*|$ -monomial if it is of the form  $t \mapsto \alpha t^n$  with  $\alpha \in |k^*|$  and  $n \in \mathbb{Z}$ . The integer  $n$  is then called the degree (or logarithmic slope) of  $f$ .

A map  $f : I \rightarrow \mathbb{R}_+$  is called piecewise  $|k^*|$ -monomial if  $I$  may be written as a finite union of intervals  $I_i$  with non-empty interiors such that the restriction of  $f$  to each  $I_i$  is  $|k^*|$ -monomial. For each  $x \in \overset{\circ}{I}$ , we may then define the left and right degrees of  $f$  at  $x$ . We denote them respectively by  $\deg_f^-(x)$  and  $\deg_f^+(x)$ . We say that a point  $x \in \overset{\circ}{I}$  is a break-point of  $f$  when  $\deg_f^-(x) \neq \deg_f^+(x)$ . It will also be convenient to declare the points at the boundary of  $I$  to be break-points. Note that, if  $f$  is continuous, then all break-points in  $\overset{\circ}{I}$  belong to  $|k^*|$ .

**Lemma 2.3** ([Tem14, Lemma 2.2.5]). *Let  $\varphi : D_2 \rightarrow D_1$  be a finite morphism of strict open discs. Let  $z \in D_2(k)$ . The function  $f_{\varphi, z}$  is continuous and piecewise  $|k^*|$ -monomial and its right degree coincides with the restriction of  $n_\varphi$  to  $l_z$  (canonically identified to  $[0, 1)$ ).  $\square$*

*Remark 2.4.* As explained above, since  $f_{\varphi, z}$  is continuous and piecewise  $|k^*|$ -monomial, the break-points of  $f_{\varphi, z}$  belong to  $|k| \cap [0, 1)$ . In other words, the points where the restriction of  $n_\varphi$  to  $l_z$  is not locally constant are of type 2.

Another related function is the following.

**Definition 2.5.** *Let  $\varphi : D_2 \rightarrow D_1$  be a finite morphism of strict open discs. For  $z \in D_1(k)$  and  $s \in (0, 1]$ , we denote by  $N_{\varphi, z}(s)$  the number of connected components of the set  $\varphi^{-1}(D(z, s^-))$ .*

**Definition 2.6** ([Tem14, 2.3.2]). *Let  $\varphi : D_2 \rightarrow D_1$  be a finite morphism of strict open discs. We say that  $\varphi$  is a radial morphism if the induced functions  $f_{\varphi,z}$  are equal for all  $z \in D_2(k)$ . In this case, the function  $f_\varphi := f_{\varphi,z}$  is called the profile function of  $\varphi$ .*

The following result is well-known and can easily be proved by studying valuation polygon and coordinate representation of the morphism.

**Lemma 2.7.** *Let  $\varphi$  be a finite morphism between open discs. Then, the image of a subdisc is a subdisc and the preimage of a subdisc is a finite union of subdiscs. Furthermore, if  $\varphi$  is radial, all the preimages have the same radius.  $\square$*

Let us now explain the relationship with  $N_{\varphi,z}$ .

**Lemma 2.8.** *Let  $\varphi : D_2 \rightarrow D_1$  be a radial finite morphism of open unit discs of degree  $d$ . The following properties hold.*

- (i) *For each  $x \in D_1$  and each  $y \in \varphi^{-1}(x)$ , we have  $\#\varphi^{-1}(x) = d/n_\varphi(y)$ .*
- (ii) *For each  $z \in D_1(k)$  and each  $r \in (0, 1)$ , we have*

$$(2) \quad N_{\varphi,z}(r) = \min(\{\#\varphi^{-1}(z_\rho) \mid \rho \in [0, r)\}) = \frac{d}{\deg_{f_\varphi}^-(f_\varphi^{-1}(r))}.$$

- (iii) *The function  $N_{\varphi,z}$  is independent of  $z \in D_1(k)$  and its points of discontinuity are exactly the break-points of  $f_\varphi^{-1}$ .*

*Proof.* Let  $x \in D_1$ . Since the morphism is radial, all points over  $x$  have the same radius. By Lemma 2.3,  $n_\varphi$  then takes the same value on all of them. Point (i) follows.

Let us now prove point (ii). Let  $z \in D_1(k)$  and let  $r \in (0, 1)$ . By Lemma 2.7, the preimage of the open disc  $D(z, r)^-$  is a disjoint union of open discs. Such an open disc contains only one point over  $z_\rho$  for  $\rho \in (0, r)$  close enough to  $r$ . The first equality follows. The second one follows from point (i) and Lemma 2.3.

Point (iii) follows from point (ii).  $\square$

As a consequence, when the morphism  $\varphi$  is radial, we often write  $N_\varphi$  instead of  $N_{\varphi,z}$ .

**Lemma 2.9.** *Let  $\varphi : D_2 \rightarrow D_1$  be a radial morphism of open unit discs with profile function  $f$ . Let  $0 = t_0 < t_1 < \dots < t_n = 1$  be the breaks of  $f$  and, for  $i = 1, \dots, n$ , let  $d_i$  denote the degree of  $f$  over the interval  $(t_{i-1}, t_i)$ . Let  $z \in D_2(k)$  and let  $T_{\varphi(z)} = \varphi(T_z) = \sum_{i \geq 0} a_i T_z^i$  be the coordinate representation of  $\varphi$  with respect to  $T_z$  and  $T_{\varphi(z)}$ . Then, for  $i = 1, \dots, n$ , we have*

$$(3) \quad |a_{d_i}| = |a_{d_1}| \prod_{j=1}^{i-1} t_j^{d_j - d_{j+1}}.$$

For each  $m = 0, \dots, n-1$  and each  $r \in (t_m, t_{m+1}]$ , we have

$$(4) \quad f(r) = |a_{d_1}| r^{d_{m+1}} \prod_{j=1}^m t_j^{d_j - d_{j+1}}.$$

In particular, for  $i = 1, \dots, n$ , we have

$$(5) \quad f(t_i) = |a_{d_1}| t_i^{d_i} \prod_{j=1}^{i-1} t_j^{d_j - d_{j+1}}.$$

*Proof.* For  $i = 0, \dots, n-1$  and  $\rho \in (t_i, t_{i+1})$ , we have  $f(\rho) = |\varphi(T_z)|_\rho = \max |a_i| \rho^i = |a_{d_{i+1}}| \rho^{d_{i+1}}$ , because  $f$  is  $|k^*|$ -monomial over  $(t_i, t_{i+1})$  of degree  $d_{i+1}$ .

Because of the continuity of  $|\varphi(T_z)|_{(\cdot)}$  at  $t_i$ , for  $i = 1, \dots, n-1$ , we deduce that  $|a_{d_i}| t_i^{d_i} = |a_{d_{i+1}}| t_i^{d_{i+1}}$ , which implies  $|a_{d_{i+1}}| = |a_{d_i}| t_i^{d_i - d_{i+1}}$ . Equation (3) then follows by induction.

Equation (4) follows by noting that  $f(r) = |\varphi(T_z)|_r = |a_{d_{m+1}}| r^{d_{m+1}}$  and using (3).  $\square$

**2.2.2. Profile functions in the general case.** Let  $\varphi: Y \rightarrow X$  be a finite morphism between quasi-smooth  $k$ -analytic curves. We denote by  $Y^{(2)}$  (resp.  $Y^{\text{hyp}}$ ) the set of type 2 (resp. non-rigid) points of  $Y$ .

For every point  $y$  in  $Y$ , we denote by  $\mathcal{D}_y$  the set of connected components of  $Y \setminus \{y\}$  whose closure contains  $y$ .

**Theorem 2.10.** *Let  $y \in Y^{(2)}$ . There exists a finite subset  $F_y$  of  $\mathcal{D}_y$  and a map  $Tf_\varphi^y: [0, 1] \rightarrow [0, 1]$  such that, for each  $D$  in  $\mathcal{D}_y \setminus F_y$ ,  $D$  and  $\varphi(D)$  are strict open discs and the induced morphism  $D \rightarrow \varphi(D)$  is radial with profile function  $Tf_\varphi^y$ .*

*Proof.* It is a consequence of the existence of a so-called “radializing skeleton”, see [Tem14, Theorem 3.3.11]. Note that this reference only handles compact strictly  $k$ -analytic curves, but this easily implies the general case.  $\square$

In [Tem14, Section 3.4.1], for each  $y \in Y^{(2)}$ , Michael Temkin defines the *profile function* of  $\varphi$  at  $y$  to be  $Tf_\varphi^y$ .

Putting together all those function gives a map  $Tf_\varphi^{(2)}: Y^{(2)} \times [0, 1] \rightarrow [0, 1]$ . By [Tem14, Theorem 3.4.8], it extends uniquely to a map  $Tf_\varphi: Y^{\text{hyp}} \times [0, 1] \rightarrow [0, 1]$  that is piecewise  $|k^\times|$ -monomial and continuous on each interval inside  $Y^{\text{hyp}}$ .

**Lemma 2.11.** *Let  $y \in Y^{\text{hyp}}$  and set  $x := \varphi(y)$ . Let  $K$  be an algebraically closed complete valued extension of  $\mathcal{H}(y)$ . Then, the connected components of  $\pi_K^{-1}(y) \setminus \{\sigma_K(y)\}$  and  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$  are strict open discs.*

*Let  $D$  be a connected component of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$ . The set  $\varphi^{-1}(D) \cap \pi_K^{-1}(y)$  is a disjoint union of connected components  $D'_1, \dots, D'_s$  of  $\pi_K^{-1}(y) \setminus \{\sigma_K(y)\}$  and, for each  $j = 1, \dots, s$ , the induced morphism  $\psi_j: D'_j \rightarrow D$  is radial. Moreover, the integer  $s$ , the functions  $f_{\psi_j}$  and  $N_{\psi_j}$  and the degree of  $\psi_j$  only depend on  $y$  (i.e. not on  $K$ ,  $D$  or  $j$ ).*

*Proof.* Up to shrinking  $Y$  and  $X$  around  $y$  and  $x$ , we may assume that  $\varphi^{-1}(x) = \{y\}$ .

By [PP15, Theorem 2.15], the connected components of  $\pi_K^{-1}(y) \setminus \{\sigma_K(y)\}$  are isomorphic to open discs with boundary  $\{y_K\}$ . Since the point  $\sigma_K(y)$  has type 2, those discs are strict. The same result holds for  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$ .

Let  $D$  be a connected component of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$ . It follows from the continuity of  $\varphi_K$  that  $\varphi_K^{-1}(D)$  is a disjoint union of connected components  $D'_1, \dots, D'_s$  of  $\pi_K^{-1}(y) \setminus \{\sigma_K(y)\}$ .

To prove that  $\psi_j$  is radial, we may extend the scalars and assume that  $K$  is maximally complete. By [PP15, Corollary 2.20], the group  $\text{Gal}^c(K/k)$  of continuous  $k$ -linear automorphisms of  $K$  acts transitively on the set of  $K$ -rational points of  $D'_j$  and the result follows.

To prove that  $\mathfrak{s}$ ,  $f_{\psi_j}$ ,  $N_{\psi_j}$  and  $\deg(\psi_j)$  do not depend on  $D$  and  $j$ , we may also assume that  $K$  is maximally complete. By *ibid.*,  $\text{Gal}^c(K/k)$  acts transitively on the set of connected components of  $\pi_K^{-1}(y) \setminus \{\sigma_K(y)\}$  and  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$  and the result follows.

It is clear that  $\mathfrak{s}$ ,  $f_{\psi_j}$ ,  $N_{\psi_j}$  and  $\deg(\psi_j)$  are independent of  $K$  since any two complete valued extensions of  $\mathcal{H}(y)$  may be embedded into a common one.  $\square$

*Remark 2.12.* The statement still holds, with the same proof, if  $k$  has positive characteristic.

**Notation 2.13.** *In the setting of Lemma 2.11, we set  $\mathfrak{s}_\varphi^y := \mathfrak{s}$ ,  $f_\varphi^y := f_{\psi_1}$ ,  $N_\varphi^y := N_{\psi_1}$  and  $\mathfrak{i}_\varphi^y := \deg(\psi_1)$ . The map  $f_\varphi^y$  will be called the profile function of  $\varphi$  at  $y$ . We may remove  $\varphi$  from the notation when it is clear from the context.*

**Lemma 2.14.** *Let  $y \in Y^{\text{hyp}}$ . Let  $L$  be an algebraically closed complete valued extension of  $\mathcal{H}(y)$ . Then, we have  $\mathfrak{s}_{\varphi_L}^{\sigma_L(y)} = \mathfrak{s}_\varphi^y$ ,  $f_{\varphi_L}^{\sigma_L(y)} = f_\varphi^y$ ,  $N_{\varphi_L}^{\sigma_L(y)} = N_\varphi^y$  and  $\mathfrak{i}_{\varphi_L}^{\sigma_L(y)} = \mathfrak{i}_\varphi^y$ .*

*Proof.* Note that  $\mathcal{H}(\sigma_L(y))$  contains  $\mathcal{H}(y)$ . Let  $K$  be an algebraically closed complete valued extension of  $\mathcal{H}(\sigma_L(y))$ . The result follows easily from the fact that  $\sigma_{K/k} = \sigma_{K/L} \circ \sigma_{L/k}$ .  $\square$

We now explain how to interpret  $\mathfrak{s}_\varphi^y$  as a separable degree and  $\mathfrak{i}_\varphi^y$  as an inseparable degree. Recall that, for each  $y \in X^{(2)}$ , the residue field  $\widetilde{\mathcal{H}(y)}$  is of transcendence degree 1 over  $\tilde{k}$ . In particular, it is the function field of a well-defined smooth connected projective curve over  $\tilde{k}$ . We denote it by  $\mathcal{C}_y$  and call it the *residue curve* at  $y$ .

**Lemma 2.15.** *Let  $y \in Y^{(2)}$ . For every complete valued extension  $K$  of  $k$ , we have a canonical isomorphism*

$$\mathcal{C}_y \otimes_{\tilde{k}} \tilde{K} \xrightarrow{\sim} \mathcal{C}_{\sigma_K(y)}.$$

*Proof.* We may assume that  $Y$  is reduced since reducing its curve leaves  $\mathcal{H}(y)$  unchanged. Let  $W = \mathcal{M}(\mathcal{B})$  be a strictly  $k$ -affinoid domain of  $Y$  whose Shilov boundary is exactly  $\{y\}$ . By [Ber90, Proposition 2.4.4 (iii)],  $\tilde{W} = \text{Spec}(\tilde{\mathcal{B}})$  is irreducible and its generic point is the reduction  $\tilde{y}$  of  $y$ . Since  $k$  is algebraically closed, by [BGR84, Proposition 6.2.1/4 (ii)], we have  $|\mathcal{B}|_{\text{sup}} = |k|$ , hence, by [Ber90, Proposition 2.4.4 (ii)], an isomorphism

$$\text{Frac}(\tilde{\mathcal{B}}) \xrightarrow{\sim} \tilde{k}(\tilde{y}) \xrightarrow{\sim} \widetilde{\mathcal{H}(y)}.$$

Since  $k$  is algebraically closed, it is stable (see [BGR84, Section 3.6] for definitions). Since  $W$  is reduced, by [BGR84, Theorem 6.4.3/1],  $W$  is distinguished. Since  $\tilde{k}$  is also

algebraically closed,  $\tilde{W}$  is geometrically integral. In particular,  $\tilde{\mathcal{B}} \otimes_{\tilde{k}} \tilde{K}$  is a domain. By [Ber90, Proposition 5.2.5 and its proof], the point  $\sigma_K(y)$  is the unique point in the Shilov boundary of  $W_K = \mathcal{M}(\mathcal{B} \hat{\otimes}_k K)$ , hence, by [Ber90, Proposition 2.4.4] again, we have an isomorphism

$$\text{Frac}(\widetilde{\mathcal{B} \hat{\otimes}_k K}) \xrightarrow{\sim} \mathcal{H}(\widetilde{\sigma_K(y)}).$$

By [Bos69, §6, Satz 4], we have an isomorphism  $\widetilde{\mathcal{B} \hat{\otimes}_k K} \simeq \tilde{\mathcal{B}} \otimes_{\tilde{k}} \tilde{K}$  and the result follows.  $\square$

Note that, for each  $y \in Y^{(2)}$ ,  $\varphi$  naturally induces a morphism  $\mathcal{H}(\varphi(y)) \rightarrow \mathcal{H}(y)$ , hence a morphism  $\widetilde{\mathcal{H}(\varphi(y))} \rightarrow \widetilde{\mathcal{H}(y)}$  and finally a morphism  $\tilde{\varphi}_y : \mathcal{C}_y \rightarrow \mathcal{C}_{\varphi(y)}$  between the residue curves.

Recall that, since  $k$  is algebraically closed, for each  $y \in X^{(2)}$ ,  $\mathcal{H}(y)$  is stable (see [Tem10, Corollary 6.3.6] or [Duc14, Théorème 3.3.16]), hence we have

$$[\mathcal{H}(y) : \mathcal{H}(\varphi(y))] = [\widetilde{\mathcal{H}(y)} : \widetilde{\mathcal{H}(\varphi(y))}] = \deg(\tilde{\varphi}_y).$$

**Lemma 2.16.** *Let  $y \in Y^{(2)}$  and set  $x := \varphi(y)$ . Let  $K$  be a complete valued extension of  $k$ . Then, the base-change of  $\tilde{\varphi}_y : \mathcal{C}_y \rightarrow \mathcal{C}_x$  to  $\tilde{K}$  coincides with  $(\varphi_K)_{\sigma_K(y)} : \mathcal{C}_{\sigma_K(y)} \rightarrow \mathcal{C}_{\sigma_K(x)}$ .*

*Let  $D$  be a connected component of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$ . Then, the number of connected components of  $\varphi_K^{-1}(D) \cap \pi_K^{-1}(y)$  is equal to the separable degree of  $\tilde{\varphi}_y$ , i.e. to the separable degree of  $\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(x)}$ .*

*For each connected component  $D'$  of  $\varphi_K^{-1}(D) \cap \pi_K^{-1}(y)$ , the degree of  $(\varphi_K)|_{D'}$  is equal to the inseparable degree of  $\tilde{\varphi}_y$ , i.e. to the inseparable degree of  $\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(x)}$ .*

*Proof.* We may assume that  $Y$  and  $X$  are reduced. As in the proof of Lemma 2.15, we consider a strictly  $k$ -affinoid domain  $W$  of  $Y$  whose Shilov boundary is exactly  $\{y\}$  and a strictly  $k$ -affinoid domain  $V$  of  $X$  whose Shilov boundary is exactly  $\{x\}$ . We may assume that  $\varphi^{-1}(x) = \{y\}$  and  $\varphi^{-1}(V) \subseteq W$ , so that  $\varphi$  induces a morphism of affinoid spaces  $\phi : W \rightarrow V$ , hence, by reduction, a morphism of affine curves  $\tilde{\phi} : \tilde{W} \rightarrow \tilde{V}$ . The later morphism extends uniquely to a morphism of projective curves  $\mathcal{C}_y \rightarrow \mathcal{C}_x$ , which is nothing but  $\tilde{\varphi}_x$ , by the proof of Lemma 2.15.

By base-changing to  $K$ , similarly, we find a morphism of affine curves  $\phi_K : \tilde{W}_K \rightarrow \tilde{V}_K$  that extends uniquely to  $\tilde{\phi}_K : (\varphi_K)_{\sigma_K(y)}$ . By the proof of Lemma 2.15 again, we have  $\tilde{W}_K = \tilde{W} \otimes_{\tilde{k}} \tilde{K}$  and  $\tilde{V}_K = \tilde{V} \otimes_{\tilde{k}} \tilde{K}$  and the first part of the result follows.

We now consider the reduction map  $V \rightarrow \tilde{V}$ . Recall that the reduction  $\tilde{x}$  of  $x$  is the generic point of the curve  $\tilde{V}$ . Let  $v$  be a closed point of  $\tilde{V}_K$  over  $\tilde{x}$ . Since  $\tilde{k}$  is algebraically closed,  $v$  is smooth, hence its preimage  $T_v$  by the reduction map  $V_K \rightarrow \tilde{V}_K$  is an open disc. It follows that  $T_v$  is a connected component of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$ .

The preimage of the generic point of  $\tilde{V}_K$  by the reduction map being  $\sigma_K(x)$  (see the proof of Lemma 2.15), we have just described all the connected components of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$ . In particular, there exists a closed point  $v$  of  $\tilde{V}_K$  over  $\tilde{x}$  such that  $T_v = D$ .

By a similar argument, for each point  $w$  in  $\tilde{W}_K$  over  $v$ , the preimage  $T_w$  of  $w$  by the reduction map  $W_K \rightarrow \tilde{W}_K$  is a connected component of  $\pi_K^{-1}(y) \setminus \{\sigma_K(y)\}$ . It follows that the connected components of  $\varphi_K^{-1}(T_v) \cap \pi_K^{-1}(y)$  are in bijection with the preimages of  $v$  in  $\tilde{W}_K$ . Since  $v$  lies over the generic point of  $\tilde{W}$ , the number of its preimages by  $\tilde{\phi}_K$  is equal to the separable degree of  $\tilde{\phi}$ , which coincides with the separable degree of  $\tilde{\varphi}_y$ .

Finally, let  $D'$  be a connected component of  $\varphi_K^{-1}(D) \cap \pi_K^{-1}(y)$ . From Lemma 2.11, it follows that  $\deg((\varphi_K)|_{D'})$  does not depend on  $D'$ . Since we have just proven that the number of connected component of  $\varphi_K^{-1}(D) \cap \pi_K^{-1}(y)$  is equal to the separable degree of  $\tilde{\varphi}_y$ , it follows that the degree of  $(\varphi_K)|_{D'}$  is equal to the degree of  $\varphi_K$  divided by the separable degree of  $\tilde{\varphi}_y$ . The result now follows from the chain of equalities

$$\deg(\varphi_K) = \deg(\varphi) = [\mathcal{H}(y) : \mathcal{H}(x)] = [\widetilde{\mathcal{H}(y)} : \widetilde{\mathcal{H}(x)}] = \deg(\tilde{\varphi}_y).$$

□

We now define the residual degree, and variants of it, of the morphism  $\varphi$  at a point  $y \in Y^{\text{hyp}}$ . Let us begin with type 2 points.

**Definition 2.17.** *Let  $y \in Y^{(2)}$ . We define the residual degree (resp. residual separable degree, resp. residual inseparable degree) of  $\varphi$  at  $y$  to be the degree (resp. separable degree, resp. inseparable degree) of the morphism  $\tilde{\varphi}_y$ , or equivalently of the extension  $\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(x)}$ .*

In order to extend the definition to arbitrary points of  $Y^{\text{hyp}}$ , note that, for each complete valued extension  $K$  of  $\mathcal{H}(y)$ , the point  $\sigma_K(y)$  is of type 2.

**Definition 2.18.** *Let  $y \in Y^{\text{hyp}}$ . Let  $K$  be an algebraically closed complete valued extension of  $\mathcal{H}(y)$ . We define the residual degree (resp. residual separable degree, resp. residual inseparable degree) of  $\varphi$  at  $y$  to that of  $\varphi_K$  at  $\sigma_K(y)$ .*

The definition does not depend on the choice of the extension  $K$ . Indeed, one may always embed two extensions  $K$  and  $K'$  into a common one  $L$  and the result then follows from Lemma 2.16 (applied first with  $y = \sigma_L(y) = \sigma_{L/K}(\sigma_K(y))$ ,  $K = L$ ,  $k = K$  and then with  $y = \sigma_L(y) = \sigma_{L/K'}(\sigma_{K'}(y))$ ,  $K = L$ ,  $k = K'$ .)

Moreover, for  $y \in Y^{(2)}$ , the definition coincides with the former by Lemma 2.16.

*Remark 2.19.* It follows from the stability of points of type 2 (see the reminder before Lemma 2.16) that, for each  $y \in Y^{\text{hyp}}$ , the residual degree of  $\varphi$  at  $y$  is equal to  $[\mathcal{H}(y) : \mathcal{H}(\varphi(y))]$ , i.e. the degree of  $\varphi$  at  $y$ .

Note that, with Definition 2.18, the result of Lemma 2.16 extends readily to points of type 3 and 4, which shows that the quantity  $\mathfrak{s}_\varphi^y$  from Notation 2.13 is nothing but the residual separable degree of  $\varphi$  at  $y$  while  $\mathfrak{i}_\varphi^y$  (in the notation of Lemma 2.11) is the residual inseparable degree of  $\varphi$  at  $y$ .

**Lemma 2.20.** *Let  $y \in Y^{\text{hyp}}$  and set  $x := \varphi(y)$ . Let  $K$  be an algebraically closed complete valued extension of  $k$ . Then, for each connected component  $D$  of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$ , the number of connected components of  $\varphi_K^{-1}(D) \cap \pi_K^{-1}(y)$  is equal to the residual separable degree of  $\varphi$  at  $y$  and, for each connected component  $D'$  of  $\varphi_K^{-1}(D) \cap \pi_K^{-1}(y)$ , the degree of  $(\varphi_K)|_{D'}$  is equal to the residual inseparable degree of  $\varphi$  at  $y$ .  $\square$*

**Proposition 2.21.** *For each  $y \in Y^{\text{hyp}}$ , we have  $f_\varphi^y = {}^T f_\varphi^y$ . In particular, the map  $f_\varphi : y \mapsto f_\varphi^y$  is piecewise  $|k^\times|$ -monomial and continuous on each interval inside  $Y^{\text{hyp}}$ .*

*Proof.* Let  $y \in Y^{(2)}$ . Let  $K$  be an algebraically closed complete valued extension of  $\mathcal{H}(y)$ . We use the notation of Theorem 2.10 for  $y$  and  $\sigma_K(y)$ .

The preimage of each element of  $\mathcal{D}_y \setminus F_y$  being a disjoint union of elements of  $\mathcal{D}_{\sigma_K(y)} \setminus F_{\sigma_K(y)}$ , there exists  $D \in \mathcal{D}_y \setminus F_y$  such that one of the connected components of  $\pi_K^{-1}(D)$  belongs to  $\mathcal{D}_{\sigma_K(y)} \setminus F_{\sigma_K(y)}$ . It follows that  ${}^T f^{\sigma_K(y)} = {}^T f^y$ .

Moreover, by Lemma 2.11, all the connected components of  $\pi_K^{-1}(y) \setminus \{\sigma_K(y)\}$  are connected components of  $Y_K \setminus \{\sigma_K(y)\}$ . Since there are infinitely many of them, one of them belongs to  $\mathcal{D}_{\sigma_K(y)} \setminus F_{\sigma_K(y)}$  and it follows that  ${}^T f^{\sigma_K(y)} = f^y$ .

Let  $y \in Y^{\text{hyp}}$ . Let  $I$  be an interval inside  $Y^{\text{hyp}}$  containing  $y$  and let  $(z_n)_{n \geq 0}$  be a sequence of points of type 2 of  $I$  converging to  $y$ . The sequence of Temkin's profile functions  $({}^T f^{z_n})_{n \geq 0}$  converges to Temkin's profile function  ${}^T f^y$ .

Let  $K$  be an algebraically closed complete valued extension of  $\mathcal{H}(y)$ . The map  $\sigma_K$  being continuous and sending intervals to intervals, as before,  ${}^T f^{\sigma_K(z_n)}$  tends to  ${}^T f^{\sigma_K(y)}$  when  $n$  tends to infinity. Each  $z_n$  being of type 2, by the result above, we have  ${}^T f^{\sigma_K(z_n)} = {}^T f^{z_n} = f^{z_n}$ . Since  $\sigma_K(y)$  is also of type 2, we have  ${}^T f^{\sigma_K(y)} = f^{\sigma_K(y)} = f^y$  by Lemma 2.14. By uniqueness of the limit, we deduce that  $f^y = {}^T f^y$ .

The remaining part of the result now follows from the similar statement for Temkin's profile function.  $\square$

*Remark 2.22.* The statement still holds, with the same proof, if  $k$  has positive characteristic. In particular, our method provides an alternative definition of the profile function at every point (see Notation 2.13) that is direct and makes no use of continuity.

We now give examples of profile functions. As before, we work with a finite morphism  $\varphi : Y \rightarrow X$  between quasi-smooth  $k$ -analytic curves.

*Example 2.23.* Let  $y \in Y^{\text{hyp}}$ . Assume that  $\varphi$  is residually separable at  $y$  (i.e. the residual separable degree of  $\varphi$  at  $y$  is equal to the residual degree of  $\varphi$  at  $y$ ). Then, we have  $f_\varphi^y = \text{id}$ .

Indeed, set  $x := \varphi(y)$  and let  $K$  be an algebraically closed complete valued extension of  $\mathcal{H}(y)$ . By Lemma 2.20 and Remark 2.19, for each connected component  $D$  of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$ , the number of connected components of  $\varphi_K^{-1}(D) \cap \pi_K^{-1}(y)$  is equal to the degree of  $\varphi$  at  $y$ , which is also the degree of  $\varphi_K$  at  $\sigma_K(y)$ . It follows that the morphism induced by  $\varphi_K$  on each of these connected components has degree one, hence is an isomorphism.

*Example 2.24.* Suppose now that  $\text{char}(\tilde{k}) = p > 0$  and let  $a \in k^*$ . Let  $\varphi$  and  $\phi : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$  be the Frobenius morphism and off-centered Frobenius morphism at  $a$ , respectively. Their action on rational points is given by  $\varphi : x \in \mathbb{A}_k^1(k) \mapsto x^p \in \mathbb{A}_k^1(k)$  and  $\phi : x \in \mathbb{A}_k^1(k) \mapsto (x + a)^p - a^p \in \mathbb{A}_k^1(k)$ . Then, a direct computation shows that, for each  $\rho > 0$  and each  $s \in [0, 1]$ , we have

$$f_\varphi^{\eta_\rho}(s) = \begin{cases} |p|s & \text{if } s \in [0, |p|^{\frac{1}{p-1}}) \\ s^p & \text{if } s \in [|p|^{\frac{1}{p-1}}, 1]. \end{cases}$$

On the other side, for  $\rho \in (0, |a| |p|^{\frac{1}{p-1}}]$  we have

$$f_\phi^{\eta_\rho}(s) = 1, \quad \text{for every } s \in [0, 1],$$

while for  $\rho \in (|a| |p|^{\frac{1}{p-1}}, |a|]$

$$f_\phi^{\eta_\rho}(s) = \begin{cases} |p| |a|^{p-1} \rho^{1-p} s & \text{if } s \in [0, |p|^{\frac{1}{p-1}} |a| \rho^{-1}) \\ s^p & \text{if } s \in [|p|^{\frac{1}{p-1}} |a| \rho^{-1}, 1]. \end{cases}$$

In particular, for  $\rho \geq |a|$  we have  $f_\phi^{\eta_\rho} = f_\varphi^{\eta_\rho}$ .

The last examples can be widely generalized. In order to do so, we will need to use the notion of the different  $\delta_{L/K}$  of an extension of one-dimensional analytic fields  $L/K$  from [CTT16]. We refer to this article for the definition and basic properties and simply give a way to compute it in the case we are interested in.

Let  $y \in Y^{\text{hyp}}$ . Recall that a tame parameter at  $y$  is an element  $t_y \in \mathcal{O}_y$  such that the extension of valued fields  $\mathcal{H}(y)/\widehat{k(t_y)}$ , where  $\widehat{k(t_y)}$  denotes the closure of  $k(t_y)$  in  $\mathcal{H}(y)$ , is tamely ramified. The element  $t_y$  induces a morphism  $\psi_y$  from some affinoid domain of  $Y$  containing  $y$  to  $\mathbb{A}^1$ . We denote by  $r(t_y)$  the radius of the point  $\psi_y(y)$ .

For each  $y \in Y^{\text{hyp}}$ , if  $t_y$  is a tame parameter at  $y$  and  $t_x$  a tame parameter at  $x := \varphi(y)$ , by [CTT16, Corollary 2.4.6], we have

$$\delta_\varphi(y) := \delta_{\mathcal{H}(y)/\mathcal{H}(x)} = \left| \frac{dt_x}{dt_y} \right| \frac{r(t_y)}{r(t_x)}.$$

**Lemma 2.25.** *Let  $y \in Y^{\text{hyp}}$ . For each complete valued extension  $K$  of  $k$ , we have*

$$\delta_{\varphi_K}(\sigma_K(y)) = \delta_\varphi(y).$$

*Proof.* Let  $K$  be a complete valued extension of  $k$ . Assume that  $y$  is of type 3 or 4. Then,  $y$  has a neighborhood isomorphic to an analytic domain of the affine line, hence there exists  $t_y \in \mathcal{O}_y$  such that  $\widehat{k(t_y)}$  is isomorphic to  $\mathcal{H}(y)$ . Then  $t_y$  is a parameter at  $y$  and its image  $t_{\sigma_K(y)}$  in  $\mathcal{H}(\sigma_K(y))$  is also a parameter at  $\sigma_K(y)$ . Note also that the explicit description of the points of the line shows that we have  $r(t_{\sigma_K(y)}) = r(t_y)$ . Arguing similarly for the point  $x$  (which has the same type as  $y$ ), the formula follows.

Let us now assume that  $y$  is of type 2. Choose a tame parameter  $t_y$  at  $y$ . It follows from Lemma 2.16 that its image  $t_{\sigma_K(y)}$  in  $\mathcal{H}(\sigma_K(y))$  is also a parameter at  $\sigma_K(y)$ . By [PP15, Lemma 2.9], we have  $\sigma_K(\psi_y(y)) = \psi_{\sigma_K(y)}(\sigma_K(y))$ , hence  $r(t_{\sigma_K(y)}) = r(t_y)$  by a computation on the line as before. Arguing similarly for  $x$ , the result follows.  $\square$

*Example 2.26.* Let  $y \in Y^{\text{hyp}}$ . Suppose now that  $\text{char}(\tilde{k}) = p > 0$  and that  $\varphi$  is residually purely inseparable of degree  $p$  at  $y$  (i.e. the residual separable degree of  $\varphi$  at  $y$  is equal to the residual degree of  $\varphi$  at  $y$  and equal to  $p$ ). Then, for each  $s \in [0, 1]$ , we have

$$f_{\varphi}^y(s) = \begin{cases} \delta_{\varphi}(y)s & \text{if } s \in [0, \delta_{\varphi}(y)^{\frac{1}{p-1}}) \\ s^p & \text{if } s \in [\delta_{\varphi}(y)^{\frac{1}{p-1}}, 1]. \end{cases}$$

Indeed, let  $K$  be an algebraically closed complete valued extension of  $\mathcal{H}(y)$ . By Lemma 2.14, we have  $f_{\varphi_K}^{\sigma_K(y)} = f_{\varphi}^y$  and by Lemma 2.25, we have  $\delta_{\varphi_K}(\sigma_K(y)) = \delta_{\varphi}(y)$ , hence, up to replacing  $y$  by  $\sigma_K(y)$ , we may assume that  $y$  is of type 2. The result then follows from Proposition 2.21 and from the similar result for Temkin's profile function, see [Tem14, Theorem 3.3.7].

### 3. $p$ -ADIC DIFFERENTIAL EQUATIONS

**3.1. Modules with an integrable connection.** Let  $X$  be a quasi-smooth  $k$ -analytic curve. By a differential equation on  $X$  we mean a pair  $(\mathcal{E}, \nabla)$  where  $\mathcal{E}$  is a locally free  $\mathcal{O}_X$ -module of finite type equipped with an integrable connection, i.e. a  $k$ -linear map  $\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes \mathcal{E}$  which satisfies the Leibniz rule

$$\nabla(f \otimes e) = df \otimes e + f\nabla(e).$$

We say that the differential equation has rank  $r$  if  $\mathcal{E}$  has rank  $r$ . We put  $\mathcal{E}^{\nabla} := \text{Ker } \nabla$  and the elements of the latter sheaf are called the horizontal sections of  $(\mathcal{E}, \nabla)$ .

If  $K/k$  is an extension of valued fields, then one may consider the pullback  $\pi_K^*(\mathcal{E}, \nabla)$ , which is a differential equation on the curve  $X_K$ .

**3.2. Multiradius of convergence.** We first define radii of convergence at rational points of discs. Let  $D$  be a strict open disc. Let  $x \in D(k)$ . Let us identify  $D$  with the open unit disc via a coordinate  $T_x : D \xrightarrow{\sim} D(0, 1^-)$  which sends  $x$  to 0. For each  $s \in (0, 1)$ , we set  $D(x, s^{\pm}) := T_x^{-1}(D(0, s^{\pm}))$ .

**Definition 3.1.** Let  $(\mathcal{E}, \nabla)$  be a differential equation of rank  $r$  on a strict open disc  $D$ . Let  $i \in \{1, \dots, r\}$ . We define the  $i^{\text{th}}$  radius of convergence of  $(\mathcal{E}, \nabla)$  at  $x$  to be

$$\mathcal{R}_i(x, (\mathcal{E}, \nabla)) := \sup \{s \in (0, 1) \mid \dim_k (H^0(D(x, s^-), (\mathcal{E}, \nabla))) \geq r - i + 1\}$$

and the multiradius of convergence of  $(\mathcal{E}, \nabla)$  at  $x$  to be

$$\mathcal{MR}(x, (\mathcal{E}, \nabla)) := (\mathcal{R}_1(x, (\mathcal{E}, \nabla)), \dots, \mathcal{R}_r(x, (\mathcal{E}, \nabla))).$$

We now define radii of convergence at a non-rational point  $X$  of a  $k$ -analytic curve  $X$ . Let  $x \in X^{\text{hyp}}$ . Let  $K$  be a complete algebraically closed extension of  $\mathcal{H}(x)$ . By Lemma 2.11, the connected components of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$  are strict open discs. Let  $x_K$  be a  $K$ -rational point above  $x$  and let  $D_x^g$  be the connected component of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$  that contains  $x_K$ .

**Definition 3.2.** Let  $(\mathcal{E}, \nabla)$  be a differential equation of rank  $r$  on  $X$ . Let  $i \in \{1, \dots, r\}$ . We define the  $i^{\text{th}}$  radius of convergence of  $(\mathcal{E}, \nabla)$  at  $x$  to be

$$\mathcal{R}_i(x, (\mathcal{E}, \nabla)) := \mathcal{R}_i(x_K, \pi^*(\mathcal{E}, \nabla)|_{D_x^g})$$

and the multiradius of convergence of  $(\mathcal{E}, \nabla)$  at  $x$  to be

$$\mathcal{MR}(x, (\mathcal{E}, \nabla)) := (\mathcal{R}_1(x, (\mathcal{E}, \nabla)), \dots, \mathcal{R}_r(x, (\mathcal{E}, \nabla))).$$

The definition above is independent of the choice of the field  $K$  and of the  $K$ -rational point  $x_K$  above  $x$  (see [PP15, Section 2.3]).

*Remark 3.3.* Our definition of the (multi)radius of convergence corresponds to what is usually called the spectral or intrinsic multiradius of convergence in the literature.

**3.3. Convergence polygons.** Let  $(\mathcal{E}, \nabla)$  be a differential equation of rank  $r$  on a quasi-smooth  $k$ -analytic curve  $X$ .

**Definition 3.4.** Let  $x \in X^{\text{hyp}}$ . For each  $i = 1, \dots, r$ , we set

$$h_i(x, (\mathcal{E}, \nabla)) := - \sum_{j=1}^i \log(\mathcal{R}_j(x, (\mathcal{E}, \nabla))).$$

The convergence polygon  $\mathcal{N}(x, (\mathcal{E}, \nabla))$  of  $(\mathcal{E}, \nabla)$  at the point  $x$  is the concave polygon with vertices

$$(0, 0), \quad (i, h_i(x, (\mathcal{E}, \nabla))), \quad i = 1, \dots, r.$$

The height of the convergence polygon at the point  $x$  is  $h(x, (\mathcal{E}, \nabla)) := h_r(x, (\mathcal{E}, \nabla))$ . Again, if no confusion can arise, we may omit writing  $(\mathcal{E}, \nabla)$ .

**Theorem 3.5.** Let  $i \in \{1, \dots, r\}$ . Then, the  $i^{\text{th}}$  radius function  $\mathcal{R}_i(\cdot, (\mathcal{E}, \nabla)) : X^{\text{hyp}} \rightarrow (0, 1]^r$  is continuous and piecewise  $|k^*|$ -monomial on each interval in  $X^{\text{hyp}}$ . Moreover, for each  $x \in X^{(2)}$ , it is monomial of degree 1 along almost every direction emanating from  $x$ .

*Proof.* For an interval corresponding to a skeleton of an annulus, this follows from [Ked10, Theorem 11.3.2]. However, instances of it were known much earlier and we refer to the Notes at the end of Section 11 in *ibid.* for a detailed account. For higher genus curves and intervals with type 2 and 3 endpoints and  $\mathcal{R}_1$  this is in [Bal10]. The rest of the cases are in [PP15, Theorem 3.6].  $\square$

Let  $x$  be a non-rational point of  $X$ ,  $\vec{t} \in T_x X$  be a tangent direction at  $x$  and  $I$  be an interval in  $X$  corresponding to  $\vec{t}$ . It follows from the previous theorem that the function  $h(x', (\mathcal{E}, \nabla))$  has a constant degree for  $x' \in I$  close enough to  $x$ . Denote this degree by  $n$ .

**Definition 3.6.** The quantity  $-n$  is called the irregularity of  $(\mathcal{E}, \nabla)$  at the point  $x$  in the direction  $\vec{t}$  and is denoted by  $\text{Irr}_{\vec{t}}(x, (\mathcal{E}, \nabla))$ . If  $x$  is clear from the context, we may also write just  $\text{Irr}_{\vec{t}}(\mathcal{E}, \nabla)$ .

In explicit terms let  $A_{\vec{t}}$  be a small enough strict open annulus in  $X$  attached to  $x$  that corresponds to  $\vec{t}$  and normalized via some coordinate  $T : A_{\vec{t}} \xrightarrow{\sim} A(0; q, 1)$  so that for the points on the skeleton  $\eta_\rho$  we have  $\eta_\rho \rightarrow x$  for  $\rho \rightarrow 1$ . We then may define the function  $I : (\log r, 0) \rightarrow \mathbb{R}$  by  $s \mapsto I(s) = h(\eta_\rho)$ , where  $s = \log \rho$ .

$$\text{Irr}_{\vec{t}}(x, (\mathcal{E}, \nabla)) := -\frac{d}{ds} I(0-).$$

It follows from Theorem 3.5, and more precisely from piecewise monomiality of the multiradius function, that irregularity is well defined.

The previous definition can be globalized in the following way. Let  $\Gamma$  be a finite subset of  $T_x X$ . Then, we define the Laplacian of  $(\mathcal{E}, \nabla)$  at  $x$  along  $\Gamma$  to be

$$\Delta_x(\Gamma, (\mathcal{E}, \nabla)) := \sum_{\vec{t} \in \Gamma} \text{Irr}_{\vec{t}}(\mathcal{E}, \nabla).$$

**3.4. Pushforward of differential equations.** Let  $\varphi : Y \rightarrow X$  be a finite étale morphism of quasi-smooth  $k$ -analytic curves. Let  $(\mathcal{E}, \nabla_{\mathcal{E}})$  be a differential equation on  $X$ .

Set  $\mathcal{F} := \varphi_*(\mathcal{E})$ . Since  $\varphi$  is étale, we have  $\Omega_Y^1 = \varphi^* \Omega_X^1$  hence, due to the projection formula, a map

$$\nabla_{\mathcal{F}} : \varphi_*(\mathcal{E}) \rightarrow \varphi_*(\Omega_Y^1 \otimes_{\mathcal{O}_Y} \mathcal{E}) \simeq \Omega_X^1 \otimes_{\mathcal{O}_X} \varphi_*(\mathcal{E}).$$

The couple  $\varphi_*(\mathcal{E}, \nabla_{\mathcal{E}}) =: (\mathcal{F}, \nabla_{\mathcal{F}})$  is a  $p$ -adic differential equation on  $Y$  called the *pushforward* of  $(\mathcal{E}, \nabla_{\mathcal{E}})$  by  $\varphi$ .

To lighten the notation, we will use the same symbol  $\nabla$  for both connections  $\nabla_{\mathcal{E}}$  and  $\nabla_{\mathcal{F}}$ , hoping that it will be clear from the context what connection it represents.

*Remark 3.7.* Note that it follows from definition above that for any analytic domain  $U$  of  $X$ , we have

$$(6) \quad H^0(\varphi^{-1}(U), (\mathcal{E}, \nabla)) \xrightarrow{\sim} H^0(U, (\mathcal{F}, \nabla)).$$

This fact will be extensively used in the next section.

## 4. THE MAIN RESULTS

**4.1. Pushforward of a connection: open discs.** Let  $\varphi : D_2 \rightarrow D_1$  be a finite étale morphism of degree  $d$  between open unit discs. Let  $(\mathcal{E}, \nabla)$  be a differential equation of rank  $r$  on  $D_2$  and let  $(\mathcal{F}, \nabla) := \varphi_*(\mathcal{E}, \nabla)$  be its pushforward on  $D_1$  by  $\varphi$ .

**Definition 4.1.** *Let  $x \in D_1(k)$  be a rational point. For  $s \in (0, 1]$ , we set*

$$\Phi_x(s, \varphi, (\mathcal{E}, \nabla)) := \dim_k H^0(\varphi^{-1}(D(x, s^-)), (\mathcal{E}, \nabla)).$$

*For an arbitrary point  $x \in D_1$ , we set*

$$\Phi_x(s, \varphi, (\mathcal{E}, \nabla)) := \Phi_{x_K}(s, \varphi_K, \pi_K^*(\mathcal{E}, \nabla)),$$

*where  $K$  is the completion of an algebraic closure of the complete residue field  $\mathcal{H}(x)$ .*

It is clear that, for each  $x \in D_1$ , the function  $\Phi_x$  is non-increasing, left-continuous and bounded by  $rd$ . In particular, it has finitely many breaks. Note also that, since, by Lutz' theorem [Lut37, Théorème IV], differential equations admit full sets solutions in the neighborhood of rational points, we have  $\Phi_x(s) = rd$  for  $s$  close enough to 0.

*Remark 4.2.* Assume that  $(\mathcal{E}, \nabla) = (\mathcal{O}^r, d^r)$  is the constant connection. In this case, for each  $x \in D_1(k)$  and every  $s \in (0, 1]$ ,  $\dim_k H^0(\varphi^{-1}(D(x, s^-)), (\mathcal{O}, d))$  is nothing but the number of connected components of  $\varphi^{-1}(D(x, s^-))$  multiplied by the rank  $r$ . We deduce that, for each  $x \in D_1$ , we have  $\Phi_x = rN_x$ .

For  $R \in [0, 1)$ , we set  $\Phi_x(R^+) := \lim_{s \rightarrow R^+} \Phi_x$ .

**Theorem 4.3.** *Let  $x \in D_1$ . We have*

$$\#\{i \in \{1, \dots, rd\} \mid \mathcal{R}_i(x, (\mathcal{F}, \nabla)) = 1\} = \Phi_x(1)$$

and, for each  $R \in (0, 1)$ ,

$$\#\{i \in \{1, \dots, rd\} \mid \mathcal{R}_i(x, (\mathcal{F}, \nabla)) = R\} = \Phi_x(R) - \Phi_x(R^+).$$

In particular, the radii of convergence of  $(\mathcal{F}, \nabla)$  at the point  $x$  that are smaller than 1 are exactly the break-points of  $\Phi_x$ .

*Proof.* Due to the definitions, we may assume that  $x$  is a  $k$ -rational point. Note that  $\mathcal{F}$  has rank  $rd$ . Following formula (3.1) and using (6), we may write

$$\begin{aligned} \mathcal{R}_i(x, (\mathcal{F}, \nabla)) &= \sup \left\{ s \in (0, 1) \mid \dim_k H^0(D(x, s^-), (\mathcal{F}, \nabla)) \geq rd - i + 1 \right\} \\ &= \sup \left\{ s \in (0, 1) \mid \dim_k H^0(\varphi^{-1}(D(x, s^-)), (\mathcal{E}, \nabla)) \geq rd - i + 1 \right\} \\ &= \sup \left\{ s \in (0, 1) \mid \Phi_x(s) \geq rd - i + 1 \right\}. \end{aligned}$$

The result follows.  $\square$

In the rest of the section, we will study manifestations of the previous theorem for particular cases of the connection  $(\mathcal{E}, \nabla)$ .

**Corollary 4.4.** *Assume that  $(\mathcal{E}, \nabla) = (\mathcal{O}, d)$  is the constant connection on  $D_2$  and let  $x \in D_1$ . Let  $0 = s_0 < s_1 < \dots < s_{n-1} < s_n = 1$  be the breaks of the function  $N_x$ . For  $i = 1, \dots, n$ , set  $n_{x,i} := N_x(s_i)$ . Then, the radii of convergence of  $\varphi_*(\mathcal{O}, d)$  at  $x$  are*

$$(7) \quad \begin{cases} \mathcal{R}_1(x, \varphi_*(\mathcal{O}, d)) = \dots = \mathcal{R}_{d-n_{x,2}}(x, \varphi_*(\mathcal{O}, d)) = s_1 ; \\ \mathcal{R}_{d-n_{x,2}+1}(x, \varphi_*(\mathcal{O}, d)) = \dots = \mathcal{R}_{d-n_{x,3}}(x, \varphi_*(\mathcal{O}, d)) = s_2 ; \\ \vdots \\ \mathcal{R}_{d-n_{x,n-1}+1}(x, \varphi_*(\mathcal{O}, d)) = \dots = \mathcal{R}_{d-1}(x, \varphi_*(\mathcal{O}, d)) = s_{n-1} ; \\ \mathcal{R}_d(x, \varphi_*(\mathcal{O}, d)) = 1. \end{cases}$$

*Proof.* The result follows directly from Theorem 4.3 and Remark 4.2. Indeed, we have

$$\Phi_x(1) = N_x(1) = 1$$

since it is, by definition, the number of connected components of  $\varphi^{-1}(D(x, 1^-)) = \varphi^{-1}(D_1) = D_2$  and, for each  $i = 1, \dots, n-1$ , we have

$$\Phi_x(s_i) - \Phi_x(s_i^+) = \Phi_x(s_i) - \Phi_x(s_{i+1}) = n_{x,i} - n_{x,i+1},$$

with  $n_{x,n} = N_x(1) = 1$  and  $n_{x,1} = N_x(s_1) = d$ .  $\square$

For the next corollary and the results that follow we will use the following operation: If  $F$  and  $F'$  are two finite ordered families of real numbers, we denote by  $F * F'$  the family obtained by concatenation and reordering. If  $n$  is a positive integer, we denote by  $F^{*n}$  the family  $F * \dots * F$  with  $n$  copies of  $F$ .

**Corollary 4.5.** *Assume that  $\varphi : D_2 \rightarrow D_1$  is radial with profile function  $f := f_\varphi$ . Let  $x \in D_1(k)$  and let  $0 = s_0 < s_1 < \dots < s_{n-1} < s_n = 1$  be the breaks of the function  $N_x$ . For  $i = 1, \dots, n$ , set  $n_{x,i} := N_x(s_i)$ . Suppose that there exist  $R_1 \leq \dots \leq R_r$  such that, for each  $y \in \varphi^{-1}(x)$ , we have*

$$\mathcal{MR}(y, (\mathcal{E}, \nabla)) = (R_1, \dots, R_r).$$

For  $i = 1, \dots, r$ , let  $m_i$  be the unique element of  $\{0, \dots, n-1\}$  such that  $s_{m_i} < f(R_i) \leq s_{m_i+1}$  and set

$$F_i := (s_1, \dots, s_1, \dots, s_{m_i}, \dots, s_{m_i}, f(R_i), \dots, f(R_i)),$$

where, for  $j = 1, \dots, m_i$ ,  $s_j$  appears  $n_{x,j} - n_{x,j+1}$  times and  $f(R_i)$  appears  $n_{x,m_i+1}$  times. Then, we have

$$(8) \quad \mathcal{MR}(x, (\mathcal{F}, \nabla)) = \bigstar_{i=1}^r F_i.$$

*Proof.* By assumption, for each  $y \in \varphi^{-1}(x)$  and each  $t \in (0, 1]$ , we have

$$\dim_k H^0(D(y, t^-), (\mathcal{E}, \nabla)) = \#\{i \in \{1, \dots, r\} \mid R_i \geq t\}.$$

Let  $s \in (0, 1]$ . We denote by  $D_1(s), \dots, D_{N_x(s)}(s)$  the open discs that are the connected components of  $\varphi^{-1}(D(x, s^-))$ . Since  $\varphi$  is radial, they all have the same radius, namely  $f^{-1}(s)$ . We deduce that

$$\begin{aligned} \Phi_x(s) &= \dim_k H^0(\varphi^{-1}(D(x, s^-)), (\mathcal{E}, \nabla)) \\ &= \sum_{j=1}^{N_x(s)} \dim_k H^0(D_j(s), (\mathcal{E}, \nabla)) \\ &= N_x(s) \cdot \#\{i \in \{1, \dots, r\} \mid R_i \geq f^{-1}(s)\}. \end{aligned}$$

It follows that the set of break-points  $B$  of the function  $\Phi_x$  is contained in

$$\{s_1, \dots, s_n, f(R_1), \dots, f(R_r)\}.$$

Noting that  $\#\{i \in \{1, \dots, r\} \mid R_i \geq f^{-1}(s)\} = 0$  if, and only if,  $s > f(R_r)$ , it follows that

$$B = \{s_1, \dots, s_{m_r}, f(R_1), \dots, f(R_r)\}.$$

Let  $b \in B$ . We will now compute the multiplicity of  $b$  in  $\mathcal{MR}(x, (\mathcal{F}, \nabla))$  using Theorem 4.3 and show that it is equal to its multiplicity in  $F := \bigstar_{i=1}^r F_i$ . To this end, remark that, for each  $b \in (0, 1)$ , we have

$$\Phi_x(b^+) = N_x(b^+) \cdot \#\{i \in \{1, \dots, r\} \mid R_i > f^{-1}(b)\},$$

where  $N_x(b^+) = \lim_{s \rightarrow b^+} N_x(s)$ .

• Assume that there exists  $u \in \{1, \dots, m_r\}$  such that  $b = s_u$  and  $b \neq f(R_v)$  for each  $v \in \{1, \dots, r\}$ .

Then, we have  $b = s_u \leq s_{m_r} < f(R_r) \leq 1$ , hence  $b$  appears in  $\mathcal{MR}(x, (\mathcal{F}, \nabla))$  with multiplicity

$$\Phi_x(b) - \Phi_x(b^+) = (n_{x,u} - n_{x,u+1}) \cdot \#\{i \in \{1, \dots, r\} \mid f(R_i) \geq s_u\}.$$

On the other hand, for each  $i \in \{1, \dots, r\}$ ,  $s_u$  appears in  $F_i$  with multiplicity  $n_{x,u} - n_{x,u+1}$  when  $s_u < f(R_i)$  (which is equivalent to  $s_u \leq f(R_i)$  in this case) and 0 otherwise. The result follows.

• Assume that there exists  $v \in \{1, \dots, r\}$  such that  $b = f(R_v) < s_{m_v+1}$ .

Then, we have  $b < 1$  and  $b$  appears in  $\mathcal{MR}(x, (\mathcal{F}, \nabla))$  with multiplicity

$$\Phi_x(b) - \Phi_x(b^+) = n_{x,m_v+1} \cdot \#\{i \in \{1, \dots, r\} \mid R_i = R_v\}.$$

On the other hand, for each  $i \in \{1, \dots, r\}$ ,  $b$  appears in  $F_i$  with multiplicity  $n_{x,m_v+1}$  if  $R_i = R_v$  and 0 otherwise. The result follows.

• Assume that there exists  $v \in \{1, \dots, r\}$  such that  $b = f(R_v) = s_{m_v+1} < 1$ .

Then,  $m_v \leq n - 2$  and  $b$  appears in  $\mathcal{MR}(x, (\mathcal{F}, \nabla))$  with multiplicity

$$\begin{aligned} \Phi_x(b) - \Phi_x(b^+) &= n_{x,m_v+1} \cdot \#\{i \in \{1, \dots, r\} \mid R_i \geq R_v\} \\ &\quad - n_{x,m_v+2} \cdot \#\{i \in \{1, \dots, r\} \mid R_i > R_v\} \\ &= (n_{x,m_v+1} - n_{x,m_v+2}) \cdot \#\{i \in \{1, \dots, r\} \mid R_i > R_v\} \\ &\quad + n_{x,m_v+1} \cdot \#\{i \in \{1, \dots, r\} \mid R_i = R_v\}. \end{aligned}$$

Let  $i \in \{1, \dots, r\}$ . If  $R_i > R_v$ , then  $b = s_{m_v+1}$  appears in  $F_i$  with multiplicity  $n_{x,m_v+1} - n_{x,m_v+2}$ . If  $R_i = R_v$ , then  $b = f(R_v)$  appears in  $F_i$  with multiplicity  $n_{x,m_v+1}$ . If  $R_i < R_v$ , then  $b$  does not appear in  $F_i$ . The result follows.

• Assume that  $b = 1$ .

Then we have  $R_r = 1$ ,  $m_r = n - 1$  and  $b$  appears in  $\mathcal{MR}(x, (\mathcal{F}, \nabla))$  with multiplicity

$$\Phi_x(1) = n_{x,n} \cdot \#\{i \in \{1, \dots, r\} \mid R_i = 1\}.$$

On the other hand, for each  $i \in \{1, \dots, r\}$ ,  $b = 1$  appears in  $F_i$  with multiplicity  $n_{x,n}$  if  $R_i = 1$  and 0 otherwise. The result follows.  $\square$

**4.2. Pushforward of a connection: the general case.** Let  $\varphi: Y \rightarrow X$  be a finite étale morphism between quasi-smooth  $k$ -analytic curves. Let  $(\mathcal{E}, \nabla)$  be a differential equation of rank  $r$  on  $Y$  and let  $(\mathcal{F}, \nabla) := \varphi_*(\mathcal{E}, \nabla)$  be its pushforward on  $X$  by  $\varphi$ . Let  $y \in Y^{\text{hyp}}$  and set  $x := \varphi(y)$ .

We will use Notation 2.13. Let  $0 = s_0^y < s_1^y < \dots < s_{n(y)-1}^y < s_{n(y)}^y = 1$  be the break-points of the function  $(f^y)^{-1}$  (or  $N^y$ ). For  $i = 1, \dots, n(y)$ , set  $n_i^y := N^y(s_i^y)$ .

Let  $R \in (0, 1]$ . Let  $m \in \{0, \dots, n(y) - 1\}$  be such that  $s_m^y < f^y(R) \leq s_{m+1}^y$ . Denote by  $F^y(R)$  the family

$$F^y(R) = (s_1^y, \dots, s_1^y, \dots, s_m^y, \dots, s_m^y, f^y(R), \dots, f^y(R)),$$

where, for each  $i = 1, \dots, m$ ,  $s_i^y$  appears  $n_i^y - n_{i+1}^y$  times and  $f^y(R)$  appears  $n_{m+1}^y$  times.

**Theorem 4.6.** *The multiradius of  $(\mathcal{F}, \nabla) = \varphi_*(\mathcal{E}, \nabla)$  at  $x$  is given by*

$$\mathcal{MR}(x, (\mathcal{F}, \nabla)) = \bigstar_{\substack{y \in \varphi^{-1}(x) \\ 1 \leq i \leq r}} F^y(\mathcal{R}_i(y, (\mathcal{E}, \nabla)))^{*\mathfrak{s}^y}.$$

*Proof.* Denote by  $y_1, \dots, y_q$  the preimages of  $x$  by  $\varphi$ . Since  $\varphi$  is finite, there exists an open neighborhood  $U$  of  $x$  in  $X$  and, for each  $j = 1, \dots, q$ , an open neighborhood  $V_j$  of  $y_j$  in  $Y$  such that

$$\varphi^{-1}(U) = \bigsqcup_{j=1}^q V_j.$$

For each  $j = 1, \dots, q$ , denote by  $\varphi_j: V_j \rightarrow U$  the morphism induced by  $\varphi$ .

Since the radii at a point only depend of the germ of curve at this point, we may assume that  $X = U$ . We deduce that

$$\mathcal{MR}(x, (\mathcal{F}, \nabla)) = \bigstar_{j=1}^q \mathcal{MR}(x, (\varphi_j)_*(\mathcal{E}, \nabla)|_{V_j}).$$

As a consequence, we may assume that  $\varphi^{-1}(x) = \{y\}$ .

Let  $K$  be an algebraically closed complete valued extension of  $\mathcal{H}(x)$ . Let  $x'$  be a  $K$ -rational point of  $X_K$  over  $x$ . Let  $D'$  be the connected component of  $\pi_K^{-1}(x) \setminus \{\sigma_K(x)\}$  containing  $x'$ . By definition, we have  $\mathcal{MR}(x, (\mathcal{F}, \nabla)) = \mathcal{MR}(x', \pi_K^*(\mathcal{F}, \nabla)|_{D'})$ .

Let  $D'_1, \dots, D'_s$  be the connected components of  $\varphi_K^{-1}(D')$ . For each  $j = 1, \dots, s$ , let  $\psi_j: D'_j \rightarrow D'$  be the morphism induced by  $\varphi_K$  and denote its degree by  $d_j$ . We have

$$\mathcal{MR}(x', (\varphi_K)_*\pi_K^*(\mathcal{E}, \nabla)|_{D'}) = \bigstar_{j=1}^s \mathcal{MR}(x', (\psi_j)_*\pi_K^*(\mathcal{E}, \nabla)|_{D'_j}).$$

By Lemma 2.11, the  $D'_j$ 's are strict open discs, the morphisms  $\psi_j$ 's are radial and all the profile functions coincide, the common function being  $f^y$ .

Let us set  $(R_1, \dots, R_r) := \mathcal{MR}^{\text{sp}}(y, (\mathcal{E}, \nabla))$ . Then, for every rational point  $y'$  in  $D'_j$  we have  $\mathcal{MR}(y', \pi_K^*(\mathcal{E}, \nabla)|_{D'_j}) = (R_1, \dots, R_r)$ . By further noting that  $s = \mathfrak{s}^y$  and using Corollary 4.5, we find

$$\mathcal{MR}(x', (\psi_j)_*\pi_K^*(\mathcal{E}, \nabla)|_{D'_j}) = \bigstar_{i=1}^r F^y(R_i),$$

hence

$$\mathcal{MR}(x, (\mathcal{F}, \nabla)) = \bigstar_{i=1}^r F^y(R_i)^{*s^y}.$$

□

*Remark 4.7.* It is possible to give a proof of Theorem 4.6 that does not use Corollary 4.5 but the simpler Corollary 4.4 about the constant connection if one is willing to decompose the differential module  $(\mathcal{E}, \nabla)$  with respect to the radii (which is quite an involved result, see [PP13, Corollary 3.6.9] or [Ked15, Proposition 2.2.9]). Indeed, with the notation of the proof above, the decomposition theorem allows to reduce to the case where  $(R_1, \dots, R_r) = (R, \dots, R)$ . Then, there are no radii at  $x$  bigger than  $f^y(R)$  and the radii smaller than  $f^y(R)$  are the same as those of the pushforward of the constant connection of rank  $r$ . This also gives a conceptual explanation of why the family of radii at  $x$  may be split into families associated to the different radii at  $y$ .

As a direct consequence, we can compute the radii of convergence in the situation of Examples 2.23, 2.24 and 2.26 by using the simple description of the corresponding profile functions.

**Corollary 4.8.** (i) *Suppose that  $\varphi$  is residually separable of degree  $d$  at  $y$  (cf. Example 2.23). Then, we have*

$$\mathcal{R}_{id-j}(x, (\mathcal{F}, \nabla)) = \mathcal{R}_i(y, (\mathcal{E}, \nabla)), \quad i = 1, \dots, r, \quad j = 0, \dots, d-1.$$

(ii) *Suppose that  $p := \text{char}(\tilde{k}) > 0$  and that  $\varphi$  is the Frobenius morphism on an annulus  $\varphi : x \in A(0; r_1, r_2) \mapsto x^p \in A(0; r_1^p, r_2)$ . Let  $\rho \in (r_1, r_2)$ . In this case, we have  $n(\eta_\rho) = 2$ ,  $s_{\eta_\rho}^1 = |p|^{-\frac{p}{p-1}}$ ,  $n_1^{\eta_\rho} = p$  and  $n_2^{\eta_\rho} = 1$ . Set  $i_0 := \max\{i \in \{1, \dots, r\} \mid \mathcal{R}_i(\eta_\rho, (\mathcal{E}, \nabla)) \leq |p|^{-\frac{p}{p-1}}\}$ . Then we have*

$$\begin{aligned} \mathcal{R}_{ip-j}(\eta_{\rho^d}, (\mathcal{F}, \nabla)) &= |p|^{-1} \mathcal{R}_i(\eta_\rho, (\mathcal{E}, \nabla)), & i = 1, \dots, i_0, \quad j = 0, \dots, p-1; \\ \mathcal{R}_i(\eta_{\rho^d}, (\mathcal{F}, \nabla)) &= |p| \cdot |p|^{-\frac{1}{p-1}} = |p|^{-\frac{p}{p-1}}, & i = i_0p + 1, \dots, (p-1)r + i_0; \\ \mathcal{R}_{(p-1)r+i}(\eta_{\rho^d}, (\mathcal{F}, \nabla)) &= \mathcal{R}_i(\eta_\rho, (\mathcal{E}, \nabla))^p, & i = i_0 + 1, \dots, r. \end{aligned}$$

(iii) *Let  $y \in Y^{\text{hyp}}$ . Suppose that  $p := \text{char}(\tilde{k}) > 0$  and that  $\varphi$  is residually purely inseparable of degree  $p$  at  $y$ . Let  $\delta := \delta_\varphi(y)$  be the different of  $\varphi$  at  $y$  (cf. Example 2.26). Set  $i_0 := \max\{i \in \{1, \dots, r\} \mid \mathcal{R}_i(y, (\mathcal{E}, \nabla)) \leq \delta^{\frac{p}{p-1}}\}$ . Then, we have*

$$\begin{aligned} \mathcal{R}_{ip-j}(x, (\mathcal{F}, \nabla)) &= \delta \mathcal{R}_i(y, (\mathcal{E}, \nabla)), & i = 1, \dots, i_0, \quad j = 0, \dots, p-1; \\ \mathcal{R}_i(x, (\mathcal{F}, \nabla)) &= \delta \cdot \delta^{\frac{1}{p-1}} = \delta^{\frac{p}{p-1}}, & i = i_0p + 1, \dots, (p-1)r + i_0; \\ \mathcal{R}_{(p-1)r+i}(x, (\mathcal{F}, \nabla)) &= \mathcal{R}_i(y, (\mathcal{E}, \nabla))^p, & i = i_0 + 1, \dots, r. \end{aligned}$$

*Remark 4.9.* Corollary 4.8 (i) can be easily proved directly as in [PP15, Lemma 3.23]. In Corollary 4.8 (ii), we recover the radii of convergence of the Frobenius pushforward from [Ked10, Theorem 10.5.1]. Note that it is not more difficult to get the off-centered version

(see [Ked10, Theorem 10.8.3]) by using the description of its profile function in Example 2.24.

To the best of our knowledge, these were the only examples of radii of convergence of pushforward differential equations known so far.

## 5. APPLICATIONS

**5.1. Herbrand function and multiradius of convergence.** In this section, we will establish a close relation between the profile and Herbrand functions of a morphism and the pushforward of a constant connection.

Let  $\varphi : Y \rightarrow X$  be a finite generically étale morphism between quasi-smooth  $k$ -analytic curves. Let  $y \in Y^{\text{hyp}}$  and set  $x := \varphi(y)$ .

**Theorem 5.1** ([Tem14, Theorem 4.5.2]). *The profile function  $f_\varphi^y$  coincides with the Herbrand function of the extension  $\mathcal{H}(y)/\mathcal{H}(x)$ .*  $\square$

The fields  $\mathcal{H}(y)$  and  $\mathcal{H}(x)$  are not discretely valued, so the Herbrand function from the previous statement is not the classical one (as in [Ser68, Chapitre IV] for instance). The one we use here has been defined by M. Temkin in [Tem14, Section 4.1] and is a sort of multiplicative version of the classical Herbrand function. In this section, we always use M. Temkin's conventions and notation.

Usually, Herbrand functions are defined for Galois extensions, but since they behave well with respect to extensions, it is possible to define them for separable non-normal extensions, so that the result above makes sense. In the Galois case however, an interpretation in terms of ramification filtration is available. Combined with Lemma 2.8 (ii), we now deduce the following result.

**Corollary 5.2.** *Assume that the extension  $\mathcal{H}(y)/\mathcal{H}(x)$  is Galois with group  $G$ . The real numbers  $0 < s_1^y < \dots < s_{n(y)-1}^y < 1$  are the jumps of the upper ramification filtration of the extension  $\mathcal{H}(y)/\mathcal{H}(x)$ .*

*For each  $i = 1, \dots, n(y)$ , we have  $n_i^y = (G : G^{s_{i-1}^y})$ .*  $\square$

Thanks to Corollary 4.4, we may now express the radii of convergence of the differential equation  $\varphi_*(\mathcal{O}, d)$  in terms of ramification data.

**Corollary 5.3.** *Assume that  $\varphi^{-1}(x) = \{y\}$  and that the extension  $\mathcal{H}(y)/\mathcal{H}(x)$  is Galois with group  $G$ . Then, the radii of convergence of the differential equation  $\varphi_*(\mathcal{O}, d)$  at  $x$  coincide with the non-zero upper ramification jumps of the extension  $\mathcal{H}(y)/\mathcal{H}(x)$ .*

*More precisely, if we denote by  $v_{-1} = 1 > v_0 > \dots > v_n > v_{n+1} = 0$  the jumps of the upper ramification filtration of the extension  $\mathcal{H}(y)/\mathcal{H}(x)$ , then we have*

$$\mathcal{MR}(x, \varphi_*(\mathcal{O}, d)) = (v_n, \dots, v_n, \dots, v_0, \dots, v_0, 1),$$

*where, for each  $j = 0, \dots, n$ ,  $v_j$  appears  $(G : G^{v_{j+1}}) - (G : G^{v_j})$  times.*

*Remark 5.4.* A higher dimensional analogue of the previous result, that is, a relation between ramification of Galois extensions of higher-dimensional analytic fields and pushforward of partial  $p$ -adic differential equations is a task that will be undertaken in the future.

**5.2. Irregularity, Laplacian and pushforward.** Let  $\varphi : Y \rightarrow X$  be a finite étale morphism of quasi-smooth  $k$ -analytic curves. Let  $y \in Y^{(2)}$  and set  $x := \varphi(y)$ .

For each tangent direction  $\vec{t}$  at  $y$  and each small enough open annulus  $A_{\vec{t}}$  in  $Y$  with an endpoint in  $y$  that corresponds to  $\vec{t}$ , the image  $A_{\vec{v}} = \varphi(A_{\vec{t}})$  is an open annulus that has an endpoint in  $x$  (that is,  $A_{\vec{v}}$  corresponds to a tangent direction  $\vec{v} \in T_x X$ ). In this way we obtain a map of tangent spaces  $\vec{\varphi}_y : T_y Y \rightarrow T_x X$ .

Furthermore,  $\varphi$  restricts to a finite étale morphism  $\varphi_{\vec{t}} : A_{\vec{t}} \rightarrow A_{\vec{v}}$ . If we choose suitable normalizing coordinates  $T : A_{\vec{t}} \xrightarrow{\sim} A(0; q, 1)$  and  $S : A_{\vec{v}} \xrightarrow{\sim} A(0; q^d, 1)$ , then  $\varphi_{\vec{t}}$  can be represented as a function  $S = \varphi(T) = T^d(1 + h(T))$ , where  $|h(T)|_{\rho} < 1$  for every  $\rho \in (q, 1)$ . Here  $d = d_{\vec{t}}$  is the degree of  $\varphi$  over  $A_{\vec{t}}$ . The morphism  $\varphi_{\vec{t}}$  being étale implies that the derivative  $\frac{dS}{dT} = \varphi'(T)$  does not have zeros in  $A_{\vec{t}}$ , so that we may write  $\varphi'(T) = aT^{\sigma}(1 + g(t))$ , where  $\sigma = \sigma_{\vec{t}} \in \mathbb{Z}$ ,  $a = a_{\vec{t}}$ ,  $0 < |a| \leq 1$  and  $|g(t)|_{\rho} < 1$  for every  $\rho \in (q, 1)$ . We put

$$\nu = \nu_{\vec{t}} := \sigma - d + 1.$$

**Lemma 5.5.** *Keep the notation as above and let  $\rho \in (q, 1) \cap |k^*|$  and  $\alpha \in A(0; q, 1)(k)$  with  $|\alpha| = \rho$ . Let  $D_{\alpha}$  be the maximal open disc in  $A_{\vec{t}}$  that contains  $\alpha$ ,  $T_{\alpha} : D_{\alpha} \xrightarrow{\sim} D(0, 1^-)$  be a normalizing coordinate sending  $\alpha$  to 0. Define similarly  $D_{\varphi(\alpha)}$  and  $T_{\varphi(\alpha)}$ . Let  $T_{\varphi(\alpha)} = \sum_{i \geq 0} a_i T_{\alpha}^i$  be the restriction of  $\varphi$  over  $D_{\alpha}$  with respect to the coordinates  $T_{\alpha}$  and  $T_{\varphi(\alpha)}$ . Then, we have  $|a_1| = |a|\rho^{\nu}$ .*

*Proof.* First of all, we note that for every point  $\beta \in A(0; q, 1)(k)$  we have  $|\frac{dS}{dT}(\beta)| = |\frac{dS}{dT}|_{|\beta|} = |a||\beta|^{\sigma}$ . In particular, for  $\beta \in D_{\alpha}(k)$  we have  $|\frac{dS}{dT}(\beta)| = |a|\rho^{\sigma}$ . The last expression does not change if we pass from  $S$  and  $T$  to any pair of coordinates on  $D_{\varphi(\alpha)}$  and  $D_{\alpha}$  that identify the latter discs with  $D(0, \rho^{d^-})$  and  $D(0, \rho^-)$ , respectively. In particular, we may choose  $T'_{\varphi(\alpha)} := T_{\varphi(\alpha)}/\varphi(\alpha)$  and  $T'_{\alpha} := T_{\alpha}/\alpha$  for such coordinates to obtain

$$\left| \frac{\varphi(\alpha)}{\alpha} \right| \left| \frac{dT'_{\varphi(\alpha)}}{dT'_{\alpha}}(\beta) \right| = \left| \frac{dT'_{\varphi(\alpha)}}{dT'_{\alpha}}(\beta) \right| = |a|\rho^{\sigma}$$

for every  $\beta \in D_{\alpha}(k)$ . The proof follows by choosing  $\beta = \alpha$  and noting that  $|\alpha| = \rho$  and  $|\varphi(\alpha)| = \rho^d$ .  $\square$

*Remark 5.6.* One can show that in the above setting  $|a|\rho^{\nu} = \delta_{\rho}$ , where  $\delta_{\rho}$  is the different of the extension  $\mathcal{H}(\eta_{\rho})/\mathcal{H}(\eta_{\rho^d})$ . For this aspect one may refer to [Boj17, Remark 5.1.1] and [CTT16, Section 4.2.4].

It follows from [Tem14, Theorem 3.3.1] and its proof that for  $q < 1$  close enough to 1, the restriction of the morphism  $\varphi_{\vec{t}}$  to any maximal open disc  $D$  in  $A_{\vec{t}}$  will be radial with

profile function depending only on the endpoint of  $D$ . In what follows we will assume that this property holds. Moreover, by further increasing  $q$  if necessary we may assume that the profile function is  $|k^*|$ -monomial over the skeleton of the annulus  $A_{\bar{t}}$  (rather than just piecewise  $|k^*|$ -monomial; see comments after Theorem 2.10).

We keep our setting from above. Let  $(\mathcal{E}, \nabla)$  be a differential equation of rank  $r$  on  $Y$  and let  $(\mathcal{F}, \nabla)$  its pushforward on  $X$ . We denote by  $(\mathcal{E}, \nabla)_{\bar{t}}$  the restriction of  $(\mathcal{E}, \nabla)$  to  $A_{\bar{t}}$  and by  $(\mathcal{F}, \nabla)_{\bar{t}}$  the pushforward of  $(\mathcal{E}, \nabla)_{\bar{t}}$  by  $\varphi_{\bar{t}}$  on  $A_{\bar{v}}$ .

**Theorem 5.7.** *For each  $\rho < 1$  close enough to 1, we have*

$$h(\eta_{\rho^d}, (\mathcal{F}, \nabla)_{\bar{t}}) = dh(\eta_{\rho}, (\mathcal{E}, \nabla)_{\bar{t}}) - r\nu \log \rho^d - rd \log |a|.$$

*In particular, we have*

$$(9) \quad \text{Irr}_{\bar{v}}((\mathcal{F}, \nabla)_{\bar{t}}) = \text{Irr}_{\bar{t}}((\mathcal{E}, \nabla)_{\bar{t}}) + r\nu.$$

*Proof.* For a point  $\eta_{\rho}$  on the skeleton of  $A_{\bar{t}}$ , let  $\varphi_{\bar{t}, \rho}$  be the restriction of  $\varphi_{\bar{t}}$  to the closed annulus  $A[0; \rho, \rho]$ , let  $f_{\rho}$  be the profile function at  $\eta_{\rho}$ , let  $0 = t_0^{\rho} < t_1^{\rho} < \dots < t_n^{\rho} = 1$  be the breakpoints of  $f_{\rho}$  and let  $\mathfrak{s}^{\eta_{\rho}} = \mathfrak{s}^{\rho}$  be the residual separable degree of  $\varphi_{\bar{t}, \rho}$  at  $\eta_{\rho}$ . We also denote by  $N_{\rho}$  the function  $N_{\varphi_{\bar{t}, \rho}, \eta_{\rho^d}}$  and by  $0 = s_0^{\rho} < s_1^{\rho} < \dots < s_n^{\rho} = 1$  its breakpoints. Finally, we set  $n_j^{\rho} := N_{\rho}(s_j^{\rho})$ , for  $j = 1, \dots, n$ . Since  $f_{\rho}$  is monomial over the intervals  $(t_{i-1}^{\rho}, t_i^{\rho})$ , its degree  $d_i^{\rho}$  there will be equal to  $d_n^{\rho}/n_i^{\rho}$ , by Lemma 2.8. The same lemma implies that  $f(t_j^{\rho}) = s_j^{\rho}$ . We also note that  $d_n$  is equal to the residual inseparable degree  $i^{\eta_{\rho}}$  of  $\varphi_{\bar{t}}$  at  $\eta_{\rho}$ , by Lemmas 2.16 and 2.20, so we have  $d_n \mathfrak{s}^{\eta_{\rho}} = d$ . We will be using this equality in what follows.

Let us fix  $\rho \in (q, 1)$ . We drop  $\rho$  in  $n_i^{\rho}, t_i^{\rho}, \dots$  to lighten the notation while advising the reader to keep in mind the dependence on  $\rho$ .

Let  $\mathcal{MR}(\eta_{\rho}, (\mathcal{E}, \nabla)) = (R_1^{\rho}, \dots, R_r^{\rho}) = (R_1, \dots, R_r)$  and let us fix  $i \in \{1, \dots, r\}$ . For some  $m$ , we have  $s_m < f_{\rho}(R_i) \leq s_{m+1}$ . Having in mind Theorem 4.6, we now compute the sum of the logarithms of the radii of the pushforward that come from the radius  $R_i$ , *i.e.* the radii  $s_1, \dots, s_1, \dots, s_m, \dots, s_m, f_{\rho}(R_i), \dots, f_{\rho}(R_i)$ , where each  $s_i$  appears  $n_i - n_{i+1}$  times and  $f_{\rho}(R_i)$  appears  $n_{m+1}$  times. Multiplying by  $\mathfrak{s}$  and summing over  $i$  will give us the desired result.

We use equations (4) and (5) to express  $f_{\rho}(R_i)$  and  $s_j$ 's in terms of  $t_j$ 's. By Lemma 2.8, for  $j = 0, \dots, n$ , we have  $n_j d_j = d_n$ . Note that, since  $\varphi$  is étale, we have  $d_1 = 1$ , hence  $n_1 = d_n$ . We obtain

$$\begin{aligned} & n_{m+1} \log f_{\rho}(R_i) + \sum_{j=1}^m (n_j - n_{j+1}) \log s_j \\ &= n_{m+1} \left( \log |a_1| + d_{m+1} \log R_i + \sum_{j=1}^m (d_j - d_{j+1}) \log t_j \right) \\ & \quad + \sum_{j=1}^m (n_j - n_{j+1}) \left( \log |a_1| + d_j \log t_j + \sum_{l=1}^{j-1} (d_l - d_{l+1}) \log t_l \right) \end{aligned}$$

$$\begin{aligned}
 &= n_{m+1} \log |a_1| + d_n \log R_i + \sum_{j=1}^m (n_{m+1} d_j - n_{m+1} d_{j+1}) \log t_j \\
 &\quad + \sum_{j=1}^m (n_j - n_{j+1}) \log |a_1| + \sum_{j=1}^m (n_j d_j - n_{j+1} d_j) \log t_j \\
 &\quad + \sum_{j=1}^m (n_j - n_{j+1}) \sum_{l=1}^{j-1} (d_l - d_{l+1}) \log t_l \\
 &= d_n \log |a_1| + d_n \log R_i + \sum_{j=1}^m (n_{m+1} d_j - n_{m+1} d_{j+1} + n_j d_j - n_{j+1} d_j) \log t_j \\
 &\quad + \sum_{l=1}^{m-1} \left( \sum_{j=l+1}^m (n_j d_l - n_j d_{l+1} - n_{j+1} d_l + n_{j+1} d_{l+1}) \right) \log t_l \\
 &= d_n \log |a_1| + d_n \log R_i + \sum_{j=1}^m (n_{m+1} d_j - n_{m+1} d_{j+1} + n_j d_j - n_{j+1} d_j) \log t_j \\
 &\quad + \sum_{l=1}^{m-1} (n_{l+1} d_l - n_{l+1} d_{l+1} - n_{m+1} d_l + n_{m+1} d_{l+1}) \log t_l \\
 &= d_n \log |a_1| + d_n \log R_i.
 \end{aligned}$$

By Lemma 5.5, we have  $|a_1| = |a|\rho^\nu$ . Multiplying by  $\mathfrak{s}$  and summing over  $i = 1, \dots, r$ , we obtain

$$h(\eta_{\rho^d}, (\mathcal{F}, \nabla)_{\vec{t}}) = -rd \log |a| - r\nu \log \rho^d + dh(\eta_\rho, (\mathcal{E}, \nabla)_{\vec{t}}),$$

which proves the first claim. Taking the derivative with respect to  $\log \rho^d$ , we obtain the rest.  $\square$

**Corollary 5.8.** *Suppose that  $\varphi^{-1}(x) = \{y\}$  and let  $(\mathcal{F}, \nabla) := \varphi_*(\mathcal{E}, \nabla)$ . Let  $\Gamma_x$  be a finite subset of  $T_x X$  and let  $\Gamma_y := \varphi_y^{-1}(\Gamma_x) \subset T_y Y$ . Then*

$$(10) \quad \Delta_y(\Gamma_y, (\mathcal{E}, \nabla)) = \Delta_x(\Gamma_x, (\mathcal{F}, \nabla)) + r \sum_{\vec{t} \in \Gamma_y} \nu_{\vec{t}}.$$

*Proof.* We note that since  $\Gamma_x$  is a finite subset of  $T_x X$ ,  $\Gamma_y$  is also finite, so we may calculate Laplacian of  $(\mathcal{E}, \nabla)$  along it.

Having in mind the additivity of (10) with respect to points in  $\Gamma_x$ , it is enough to prove the result when  $\Gamma_x$  is a singleton, say  $\{\vec{v}\}$ . Let  $\vec{\varphi}_y^{-1}(\vec{v}) = \{\vec{t}_1, \dots, \vec{t}_l\}$ . For each  $i = 1, \dots, l$ , let  $A_{\vec{t}_i}(0; q_i, 1)$  be a normalized open annulus in  $Y$  such that  $\varphi$  restricts to a finite étale morphism  $\varphi_i : A_{\vec{t}_i}(0; q_i, 1) \rightarrow A_{\vec{v}}(0; q, 1)$ . By increasing  $q$  if necessary, we may assume that, for each  $i = 1, \dots, l$  and each  $\rho \in (q, 1)$ , the preimage of the point  $\eta_\rho \in A_{\vec{v}}(0; q, 1)$  by  $\varphi_i$  is the point  $\eta_{\rho^{\alpha_i}} \in A_{\vec{t}_i}(0; q_i, 1)$ , where  $\alpha_i = d_{\vec{t}_i}^{-1}$ , and that the conclusion of Theorem 5.7 holds. We have

$$h(\eta_\rho, (\mathcal{F}, \nabla)) = \sum_{i=1}^l h(\eta_{\rho^{\alpha_i}}, (\mathcal{F}, \nabla)_{\vec{t}_i})$$

$$= \sum_{i=1}^l \left( d_{\vec{t}_i} h(\eta_{\rho^{\alpha_i}}, (\mathcal{E}, \nabla)_{\vec{t}_i}) - r \nu_{\vec{t}_i} \log \rho - r d_{\vec{t}_i} \log |a_{\vec{t}_i}| \right).$$

By taking derivatives with respect to  $\log \rho$ , the result follows.  $\square$

We now derive explicit formulas for Laplacians. Let us first recall the case of  $\mathbb{P}^1$ .

**Proposition 5.9** ([PP13, Proposition 6.2.11]). *Let  $(\mathcal{F}, \nabla)$  be a differential equation of rank  $r$  on an open subset  $U$  of  $\mathbb{P}^1$ . Let  $x \in U$  be a point of type 2. Let  $\Gamma_x$  be a finite subset of  $T_x X$  with at least two elements. Then, we have*

$$\Delta_x(\Gamma_x, (\mathcal{F}, \nabla)) \leq (\#\Gamma_x - 2)r$$

with equality if  $\mathcal{R}_r(x, (\mathcal{F}, \nabla)) < 1$ .

We now deduce the general case thanks to the pushforward formula. Let  $X$  be a quasi-smooth  $k$ -analytic curve and let  $(\mathcal{E}, \nabla)$  be a differential equation of rank  $r$  on  $X$ . Recall that, by Theorem 3.5, for each point  $x \in X$ , the set

$$\Gamma_x^{\text{Irr}} = \{\vec{t} \in T_x X \mid \text{Irr}_{\vec{t}}((\mathcal{E}, \nabla)) \neq 1\}$$

is finite.

We also recall the local Riemann-Hurwitz formula. Let  $X'$  be a quasi-smooth  $k$ -analytic curve and let  $\varphi : X \rightarrow X'$  be an étale morphism. Let  $x \in X$  be an inner point of type 2 such that  $x' := \varphi(x)$  is inner too. Then, by [CTT16, Theorem 4.5.4] or [Boj17, Theorem 5.3.1] and Remark 5.6, the set

$$\Gamma_x^\nu = \{\vec{t} \in T_x X \mid \nu_{\vec{t}} \neq 1 - d_{\vec{t}}\}$$

is finite and we have

$$2g(x) - 2 = d_x(2g(x') - 2) + \sum_{\vec{t} \in T_x X} (\nu_{\vec{t}} + d_{\vec{t}} - 1),$$

where  $d_x$  denotes the local degree of  $\varphi$  at  $x$ .

**Corollary 5.10.** *Let  $x \in X$  be an inner point of type 2. Then, for each finite subset  $\Gamma_x$  of  $T_x X$  containing  $\Gamma_x^{\text{Irr}}$ , we have*

$$\Delta_x(\Gamma_x, (\mathcal{E}, \nabla)) \leq (2g(x) - 2 + \#\Gamma_x)r$$

with equality if  $\mathcal{R}_r(x, (\mathcal{E}, \nabla)) < 1$ .

*Proof.* First note that, if the result holds for some finite subset  $\Gamma_x$  of  $T_x X$  containing  $\Gamma_x^{\text{Irr}}$ , then it holds for all of them.

Let  $\varphi : V \rightarrow U$  be an étale morphism of degree  $d$  from an open neighborhood of  $x$  in  $V$  to an open subset of  $\mathbb{P}^1$ . Set  $x' := \varphi(x)$ . We may assume that  $\varphi^{-1}(x') = \{x\}$ , so that  $d_x = d$ . Let  $\Gamma_{x'}$  be a finite subset of  $T_{x'} \mathbb{P}^1$  such that  $\Gamma_x := \varphi^{-1}(\Gamma_{x'})$  contains  $\Gamma_x^{\text{Irr}} \cup \Gamma_x^\nu$ .

By the local Riemann-Hurwitz formula, we have

$$\sum_{\vec{t} \in \Gamma_x} (\nu_{\vec{t}} + d_{\vec{t}} - 1) = \sum_{\vec{t} \in T_x X} (\nu_{\vec{t}} + d_{\vec{t}} - 1) = 2g(x) - 2 + 2d.$$

Moreover, we have

$$\sum_{\vec{t} \in \Gamma_x} d_{\vec{t}} = \sum_{\vec{v} \in \Gamma'_x} \sum_{\vec{t} \in \vec{\varphi}_x^{-1}(\vec{v})} d_{\vec{t}} = d \cdot \#\Gamma'_{x'},$$

hence

$$\sum_{\vec{t} \in \Gamma_x} (d_{\vec{t}} - 1) = d \cdot \#\Gamma'_{x'} - \#\Gamma_x$$

and

$$\sum_{\vec{t} \in \Gamma_x} \nu_{\vec{t}} = 2g(x) - 2 + 2d - d \cdot \#\Gamma'_{x'} + \#\Gamma_x.$$

Noting that the rank of  $\varphi_*(\mathcal{E}, \nabla)$  is  $rd$ , the result now follows from Corollary 5.8 and Proposition 5.9, together with Theorem 4.6 to show that  $\mathcal{R}_{rd}(x', \varphi_*(\mathcal{E}, \nabla)) < 1$  when  $\mathcal{R}_r(x, (\mathcal{E}, \nabla)) < 1$ .  $\square$

By a classical decomposition argument, a more general result follows. We refer to the proof of [PP13, Proposition 6.2.20] for details.

For  $x \in X^{\text{hyp}}$  and  $i = 1, \dots, r$ , we define an  $i^{\text{th}}$  Laplacian  $\Delta_x^i(\Gamma_x, (\mathcal{E}, \nabla))$  by considering the slopes of the  $i^{\text{th}}$  partial height of the Newton polygon instead of that of its total height. Denote by  $\Gamma_x^{\mathcal{R}}$  the subset of  $T_x X$  defined by the condition that, for each  $\vec{t} \in \Gamma_x^{\mathcal{R}}$ , there exists  $j \in \{1, \dots, r\}$  such that the slope of the  $j^{\text{th}}$  radius of convergence along  $\vec{t}$  is different from 1. By Theorem 3.5, it is finite.

**Corollary 5.11.** *Let  $x \in X$  be an inner point of type 2. Let  $i \in \{1, \dots, r\}$ . Then, for each finite subset  $\Gamma_x$  of  $T_x X$  containing  $\Gamma_x^{\mathcal{R}}$ , we have*

$$\Delta_x^i(\Gamma_x, (\mathcal{E}, \nabla)) \leq (2g(x) - 2 + \#\Gamma_x) i$$

with equality in the following two cases:

- a)  $i = r$  and  $\mathcal{R}_r(x, (\mathcal{E}, \nabla)) < 1$ ;
- b)  $i < r$  and  $\mathcal{R}_i(x, (\mathcal{E}, \nabla)) < \mathcal{R}_{i+1}(x, (\mathcal{E}, \nabla))$ .

*Remark 5.12.* This result appears in [Ked15, Theorem 5.3.6], although with some details left out in the proof and using a quite dense argument.

In [PP13, Proposition 6.2.20], the formula was proved under some assumptions, which are satisfied when  $g(x) = 0$  or  $\text{char}(\tilde{k}) \neq 2$ . The idea is to construct a morphism  $\varphi$  as in the proof such that the residue morphism  $\tilde{\varphi}_x : \mathcal{C}_x \rightarrow \mathcal{C}_{x'}$  is everywhere tamely ramified. When  $\text{char}(\tilde{k}) \neq 2$ , a result of Fulton shows that this is always possible (see [Ful69, Proposition 8.1]).

Here, we prove the formula in full generality and more directly, without relying on heavy geometric input.

*Remark 5.13.* In the proof of the above results we used the local Riemann-Hurwitz formula. We indicate that, conversely, this formula can be deduced from Theorem 5.7 and Corollary

5.8, combined with a suitable  $p$ -adic index theorem that relates the Euler-Poincaré characteristic of a  $p$ -adic differential equation to that of the curve itself *via* the irregularities of the equation. This aspect will be further explored in some other paper.

## REFERENCES

- [Bal10] Francesco Baldassarri. Continuity of the radius of convergence of differential equations on  $p$ -adic analytic curves. *Invent. Math.*, 182(3):513–584, 2010.
- [Bal13] Francesco Baldassarri. Radius of convergence of  $p$ -adic connections and the  $p$ -adic Rolle theorem. *Milan J. Math.*, 81(2):397–419, 2013.
- [Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
- [BGR84] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.
- [BK] Francesco Baldassarri and Kiran Kedlaya. Harmonic functions attached to meromorphic connections on non-archimedean curves. In preparation.
- [Boj17] Velibor Bojković. Riemann-Hurwitz formula for finite morphisms of  $p$ -adic curves. *Math. Zeit.*, 2017. To appear.
- [Bos69] S. Bosch. Orthonormalbasen in der nichtarchimedischen Funktionentheorie. *Manuscr. Math.*, 1:35–57, 1969.
- [CTT16] Adina Cohen, Michael Temkin, and Dmitri Trushin. Morphisms of Berkovich curves and the different function. *Adv. Math.*, 303:800–858, 2016.
- [Duc14] Antoine Ducros. La structure des courbes analytiques. Manuscript available at <http://www.math.jussieu.fr/~ducros/trirss.pdf>, 2014.
- [Ful69] William Fulton. Hurwitz schemes and irreducibility of moduli of algebraic curves. *Ann. of Math. (2)*, 90:542–575, 1969.
- [Ked10] Kiran S. Kedlaya.  *$p$ -adic differential equations*, volume 125 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.
- [Ked15] Kiran S. Kedlaya. Local and global structure of connections on nonarchimedean curves. *Compos. Math.*, 151(6):1096–1156, 2015.
- [Lut37] Elisabeth Lutz. Sur l'équation  $y^2 = x^3 - ax - b$  dans les corps  $p$ -adiques. *J. Reine Angew. Math.*, 177:238–247, 1937.
- [Poi13] Jérôme Poineau. Les espaces de Berkovich sont angéliques. *Bull. Soc. Math. France*, 141(2):267–297, 2013.
- [PP13] Jérôme Poineau and Andrea Pulita. The convergence Newton polygon of a  $p$ -adic differential equation III: Global decomposition and controlling graphs. arXiv, 2013. <http://arxiv.org/abs/1308.0859>.
- [PP15] Jérôme Poineau and Andrea Pulita. The convergence Newton polygon of a  $p$ -adic differential equation. II: Continuity and finiteness on Berkovich curves. *Acta Math.*, 214(2):357–393, 2015.
- [Pul15] Andrea Pulita. The convergence Newton polygon of a  $p$ -adic differential equation. I: Affinoid domains of the Berkovich affine line. *Acta Math.*, 214(2):307–355, 2015.
- [Ser68] Jean-Pierre Serre. *Corps locaux*. Hermann, Paris, 1968. Deuxième édition, Publications de l'Université de Nancago, No. VIII.
- [Tem10] Michael Temkin. Stable modification of relative curves. *J. Algebraic Geom.*, 19(4):603–677, 2010.
- [Tem14] Michael Temkin. Metric uniformization of morphisms of Berkovich curves. arXiv, 2014. <http://arxiv.org/abs/1410.6892>.

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