The convergence Newton polygon of a p-adic differential equation II: Continuity and finiteness on Berkovich curves

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Abstract

We study the variation of the convergence Newton polygon of a differential equation along a smooth Berkovich curve over a non-archimedean complete valued field of characteristic 0. Relying on work of the second author who investigated its properties on affinoid domains of the affine line, we prove that its slopes give rise to continuous functions that factorize by the retraction through a locally finite subgraph of the curve.

1. Introduction

Let K be a non-archimedean complete valued field of characteristic 0. Let X be a quasi-smooth K-analytic curve, in the sense of Berkovich theory. Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module of finite type endowed with an integrable connection ∇.

Let x be a K-rational point of X. By the implicit function theorem, in the neighbourhood of the point x, the curve is isomorphic to a disc, and we may consider the radius of the biggest disc on which a given horizontal section of $(\mathscr{F}, \nabla)$ (*i.e.* a given solution of the associated differential equation) converges. Considering the radii associated to the elements of a basis of horizontal sections, we define a tuple $\boldsymbol{\mathcal{R}}(x)$ that we call multiradius of convergence at x. The logarithms of its coordinates are precisely the slopes of the convergence Newton polygon at x. Since any point of the curve may be changed into a rational one by a suitable extension of the scalars, we may actually extend the definition of the multiradius of convergence to the whole curve.

It is already interesting to consider the radius of convergence $\mathcal{R}(x)$ at a point x, *i.e.* the radius of the biggest disc on which $(\mathscr{F}, \nabla)$ is trivial, or, equivalently, the minimal radius that appears in the multiradius. In [Bal10], F. Baldassarri proved the continuity of this function.

In the article [Pul12], using different methods, the second author thoroughly investigated the multiradius of convergence $\boldsymbol{\mathcal{R}}$ in the case where X is an affinoid domain of the affine line. He proved that this function is continuous and satisfies a strong finiteness property: it factorizes by the retraction through a finite subgraph of X.

In this paper, we prove that both properties extend to arbitrary quasi-smooth K-analytic curves (see theorem 3.1.2 and remark 3.1.3 for a more precise statement):

Theorem 1.1. *The map $\boldsymbol{\mathcal{R}}$ is continuous on X and factorizes by the retraction through a locally finite subgraph Γ of X.*

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Let us now give an overview of the contents of the article. We have just explained the rough idea underlying the definition of the multiradius of convergence on the curve X. Obviously, some work remains to be done in order to put together the different local normalisations in a coherent way and give a proper definition of this multiradius. This will be the content of our first section. The first definition was actually given by F. Baldassarri in [Bal10], using a semistable formal model of the curve, which led to some restrictions. Here, we will use A. Ducros's notion of triangulation, which is a way of cutting a curve into pieces that are isomorphic to virtual discs or annuli. Among other advantages, it lets us define everything we need within the realm of analytic spaces. We will explain how our radius of convergence relates to that of [Bal10] and to the usual one in the case of analytic domains of the affine line, as defined in [BDV08] or [Pul12], for instance.

The second section is devoted to the proof of the continuity and finiteness property of the multiradius of convergence $\boldsymbol{\mathcal{R}}$. The basic idea is to represent the curve X locally as a finite étale cover of an affinoid domain W of the affine line, apply the results of [Pul12] to W and pull them back. We will first present the geometric results we need (which essentially come from A. Ducros's manuscript [Duc]) to find a nice presentation of the curve, then prove a weak version of the finiteness property (local constancy of $\boldsymbol{\mathcal{R}}$ outside a locally finite subgraph) and finally the continuity (which will yield the stronger finiteness property).

Setting 1.2. For the rest of the article, we fix the following: K is a complete valued field of characteristic 0, p is the characteristic exponent of its residue field $\tilde{K}$ (either 1 or a prime number), X is a quasi-smooth K-analytic curve[1], $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite type endowed with an integrable connection ∇.

From section 2.2 on, we will assume that the curve X is endowed with a weak triangulation S.

From section 3.2 on, we will assume that the field K is algebraically closed.

2. Definitions

In this section, we define the radius of convergence of $(\mathscr{F}, \nabla)$ at any point of the curve X. To achieve this task, we will need to understand precisely the geometry of X. Our main tool will be A. Ducros's notion of triangulation (see [Duc, 4.1.13]) that we recall here. The original definition, introduced by F. Baldassarri in [Bal10], in a slightly more restrictive setting, made use of a semistable model of the curve. The two points of view are actually very close (see section 2.3.1).

2.1 Triangulations

Let us first recall that a connected analytic space is called a virtual open disc (resp. annulus) if it becomes isomorphic to a union of open discs (resp. annuli) over an algebraically closed valued field (see [Duc, 2.2.62]).

Definition 2.1.1. A subset S of X is said to be a weak triangulation of X if

i) S is locally finite and only contains points of type 2 or 3;

ii) any connected component of $X \setminus S$ is a virtual open disc or annulus.

The union of the skeletons of the connected components of $X \setminus S$ that are virtual annuli forms a locally finite graph, which is called the skeleton Γ_S of the weak triangulation S.

[1]Quasi-smooth means that Ω_X is locally free, see [Duc, 2.1.8]. This corresponds to the notion called "rig-smooth" in the rigid analytic setting.

Usually the skeleton is the union of the segments between the points of S but beware that some strange behaviours may appear: the skeleton of an open annulus endowed with the empty weak triangulation is an open segment!

Let us also point out that the skeleton of an open disc endowed with the empty weak triangulation is empty, which sometimes forces to handle this case separately.

Remark 2.1.2. Let Z be a connected component of X such that $Z \cap \Gamma_S \neq \emptyset$ (which is always the case except if Z is an open disc such that $Z \cap S = \emptyset$). Then we have a natural continuous retraction $Z \to Z \cap \Gamma_S$.

Remark 2.1.3. A. Ducros's definition is actually stronger than ours since he requires the connected components of $X \setminus S$ to be relatively compact. For example, the empty weak triangulation of the open disc is not allowed.

This property allows him to have a natural continuous retraction $X \to \Gamma_S$ defined everywhere, and also to associate to S a nice formal model of the curve X (see [Duc, section 5.3], for details).

The existence of a triangulation is one of the main results of A. Ducros's manuscript (see [Duc, théorème 4.1.14]).

THEOREM 2.1.4 (A. Ducros). *Any quasi-smooth K-analytic curve admits a triangulation, hence a weak triangulation.*

We would now like to study the effect of extending the scalars on a given weak triangulation. Let K^a be the completion of an algebraic closure of K. It is easy to check that the preimage $S_{K^a} = \pi_{K^a}^{-1}(S)$ is a weak triangulation of X_{K^a}.

To go further, let us introduce the notion of universal point of X: roughly speaking, it is a point that may be canonically lifted to any base field extension X_L of X. We refer to [Ber90, section 5.2][2] and [Poi12] for more information.

DEFINITION 2.1.5. A point x of X is said to be universal if, for any complete valued extension L of K, the tensor norm on the algebra $\mathscr{H}(x)\hat{\otimes}_K L$ is multiplicative. In this case, it defines a point of X_L that we denote by x_L.

Remark 2.1.6. By [Poi12, corollaire 3.14], over an algebraically closed field, any point is universal. This gives a way to extend the triangulation to X_L, for any complete valued extension L of K^a.

NOTATION 2.1.7. Let L be a complete valued extension of K. We denote $\mathrm{Gal}^c(L/K)$ the group of continuous automorphisms of L that induce the identity on K.

The following lemma follows from the very definition of a universal point.

LEMMA 2.1.8. *Let x be a universal point of X and L be a complete valued extension of K. For any $\sigma \in \mathrm{Gal}^c(L/K)$, we have $\sigma(x_L) = x_L$.*

In the sequel, the group $\mathrm{Gal}^c(L/K)$ will appear several times. The following lemma, which is due to B. Dwork and P. Robba (see [DR77, lemma 8.2]), will prove very useful.

LEMMA 2.1.9. *Let M be a non-archimedean valued field which is algebraically closed and maximally complete. Let L be a complete subfield of M. Any continuous automorphism of L may be extended to an automorphism of M.*

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[2] In this reference, where universal points first appeared, they are called "peaked points".

Let us now explain how to extend a weak triangulation.

Definition 2.1.10. Let L be a complete valued extension of K. Let M be an algebraically closed and maximally complete valued field that contains L and K^a isometrically. We set

$$S_L = \pi_{M/L}(\{x_M, x \in S_{K^a}\})$$

and

$$\Gamma_{S_L} = \pi_{M/L}(\{x_M, x \in \Gamma_{S_{K^a}}\}).$$

By lemmas 2.1.8 and 2.1.9, the sets S_L and Γ_{S_L} are well-defined and independent of the choices.

Lemma 2.1.11. *Let L be a complete valued extension of K. The sets S_L and Γ_{S_L} are invariant under the action of* $\mathrm{Gal}^c(L/K)$.

To extend the weak triangulation, the key point is the following result.

Theorem 2.1.12 (A. Ducros). *Let x be a point of type 2 or 3 of X_{K^a} and let L be a complete valued extension of K. The open subset $\pi^{-1}_{L/K^a}(x) \setminus \{x_L\}$ of X_L is a disjoint union of virtual open discs.*

Corollary 2.1.13. *For any complete valued extension L of K, the set S_L is a weak triangulation of X_L and his skeleton is Γ_{S_L}.*

Corollary 2.1.14. *Let M be an algebraically closed and maximally complete valued extension of K. Let x be a point of S. The set $\pi^{-1}_{M/K}(x) \setminus S_M$ is a disjoint union of open discs which are transitively permuted by the action of the Galois group* $\mathrm{Gal}^c(M/K)$.

2.2 Radius of convergence

For the rest of the article, we assume that X is endowed with a weak triangulation S.

Definition 2.2.1. Let $x \in X$. Let L be a complete valued extension of K such that X_L contains an L-rational point $\tilde{x}$ over x. We denote $D(\tilde{x}, S_L)$ the biggest open disc centred at $\tilde{x}$ that is contained in $X_L \setminus S_L$, *i.e.* the connected component of $X_L \setminus \Gamma_{S_L}$ that contains $\tilde{x}$.

Remark 2.2.2. Assume that $x \notin \Gamma_S$. In this case, the connected component C of $X \setminus \Gamma_S$ that contains x is a virtual disc and $D(\tilde{x}, S_L)$ is the connected component of C_L that contains $\tilde{x}$.

Assume that $x \in \Gamma_S$. In this case, $D(\tilde{x}, S_L)$ is the biggest open disc centred in $\tilde{x}$ that is contained in $\pi_L^{-1}(x)$.

In particular, the definition of the disc $D(\tilde{x}, S_L)$ depends only on the skeleton and not on the triangulation itself.

The following lemma is an easy consequence of the definitions.

Lemma 2.2.3. *Let M/L be an extension of complete valued fields over K. Let $\tilde{x}$ be an L-rational point of X_L. It naturally gives rise to an M-rational point $\tilde{x}_M$ of X_M. Then we have a natural isomorphism*

$$D(\tilde{x}, S_L)\hat{\otimes}_L M \xrightarrow{\sim} D(\tilde{x}_M, S_M).$$

The next one follows from lemma 2.1.9.

Lemma 2.2.4. *Let M be an algebraically closed and maximally complete valued extension of K. Let $\tilde{x}$ and $\tilde{x}'$ in X_M that project onto the same point on X. There exists $\sigma \in \mathrm{Gal}^c(M/K)$ that send $\tilde{x}$ to $\tilde{x}'$ and induces an isomorphism $D(\tilde{x}, S_M) \xrightarrow{\sim} D(\tilde{x}', S_M)$.*

Proof. By lemma 2.1.9, there exists $\sigma \in \mathrm{Gal}^c(M/K)$ that send $\tilde{x}$ to $\tilde{x}'$. By lemma 2.1.11, the skeleton S_M is invariant under $\mathrm{Gal}^c(M/K)$. Hence, the isomorphism ψ_σ of X_M induced by σ sends the disc $D(\tilde{x}, S_M)$ to a disc that contains $\tilde{x}'$ and does not meet S_M. We deduce that $\psi_\sigma(D(\tilde{x}, S_M)) \subset D(\tilde{x}', S_M)$. Using the same argument with σ^{-1}, one shows the reverse inclusion. □

In the introduction, we explained that the radius of convergence was to appear as the radius of some disc.

Notation 2.2.5. Let $\mathbf{A}^{1,\mathrm{an}}_K$ be the affine analytic line with coordinate t. Let L be a complete valued extension of K and $c \in L$. For $R > 0$, we set

$$D^+_L(c, R) = \big\{x \in \mathbf{A}^{1,\mathrm{an}}_L \,\big|\, |(t-c)(x)| \leqslant R\big\}$$

and

$$D^-_L(c, R) = \big\{x \in \mathbf{A}^{1,\mathrm{an}}_L \,\big|\, |(t-c)(x)| < R\big\}.$$

For R_1, R_2 such that $0 < R_1 \leqslant R_2$, we set

$$C^+_L(c; R_1, R_2) = \big\{x \in \mathbf{A}^{1,\mathrm{an}}_L \,\big|\, R_1 \leqslant |(t-c)(x)| \leqslant R_2\big\}.$$

For R_1, R_2 such that $0 < R_1 < R_2$, we set

$$C^-_L(c; R_1, R_2) = \big\{x \in \mathbf{A}^{1,\mathrm{an}}_L \,\big|\, R_1 < |(t-c)(x)| < R_2\big\}.$$

Unfortunately, the radius of a disc is not invariant by isomorphism. This leads us to define the radius of convergence as a relative radius inside a fixed bigger disc. The lemma that follows will help showing it is well-defined.

Lemma 2.2.6. *Let $R_1, R_2 > 0$. Up to a translation of the coordinate t, any isomorphism $\alpha : D^-_K(0, R_1) \xrightarrow{\sim} D^-_K(0, R_2)$ is given by a power series of the form*

$$f(t) = \sum_{i \geqslant 1} a_i t^i \in K[\![t]\!],$$

with $|a_1| = R_2/R_1$ and, for every $i \geqslant 2$, $|a_i| \leqslant R_2/R_1$. In particular, it multiplies distances by the constant factor R_2/R_1: for any complete valued extension L of K, we have

$$\forall x, y \in D^-_K(0, R_1)(L),\ |\alpha(x) - \alpha(y)| = \frac{R_2}{R_1}\,|x - y|.$$

As a consequence, such an isomorphism may only exist when $R_2/R_1 \in |K^*|$.

We may now adapt the usual definition of radius of convergence (see [Pul12, section 4.3 and remark 7.5], as well as [Ked10, notation 11.3.1 and definition 11.9.1]).

Definition 2.2.7. Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module of finite type with an integrable connection ∇. Let x be a point in X and L be a complete valued extension of K such that X_L contains an L-rational point $\tilde{x}$ over x. Let us consider the pull-back $(\tilde{\mathscr{F}}, \tilde{\nabla})$ of $(\mathscr{F}, \nabla)$ on $D(\tilde{x}, S_L) \simeq D^-_L(0, R)$. Let $r = \mathrm{rk}(\mathscr{F}_x)$. For $i \in [\![1, r]\!]$, we denote $\mathcal{R}'_{S,i}(x, (\mathscr{F}, \nabla))$ the radius of the maximal open disc centred at 0 on which the connection $(\tilde{\mathscr{F}}, \tilde{\nabla})$ admits at least $r - i + 1$ horizontal sections which are linearly independent over L. Let us define the i^{th} radius of convergence of $(\mathscr{F},\nabla)$ at x by $\mathcal{R}_{S,i}(x,(\mathscr{F},\nabla)) = \mathcal{R}'_{S,i}(x,(\mathscr{F},\nabla))/R$ and the multiradius of convergence of $(\mathscr{F},\nabla)$ at x by

$$\boldsymbol{\mathcal{R}}_S(x,(\mathscr{F},\nabla)) = (\mathcal{R}_{S,1}(x,(\mathscr{F},\nabla)),\dots,\mathcal{R}_{S,r}(x,(\mathscr{F},\nabla))).$$

Remark 2.2.8. With the previous notations, one may also consider the radius of the maximal open disc centred at 0 on which the connection $(\tilde{\mathscr{F}},\tilde{\nabla})$ is trivial, as in [Bal10, definition 3.1.7], for instance. This way, one recovers the radius $\mathcal{R}_{S,1}$.

Definition 2.2.7 is independent of the choices made and invariant by base-change, thanks to the preceding lemmas (first prove the independence of the isomorphism $D(\tilde{x},S_L) \simeq D^-_L(0,R)$ and in particular of R, then the invariance by base-change for rational points and finally reduce to the case where L is algebraically closed and maximally complete). We state the following for future reference.

Lemma 2.2.9. *Let L be a complete valued extension of K. For any $x \in X_L$, we have*

$$\boldsymbol{\mathcal{R}}_{S_L}(x,\pi_L^*(\mathscr{F},\nabla)) = \boldsymbol{\mathcal{R}}_S(\pi_L(x),(\mathscr{F},\nabla)).$$

Let us now explain how the function behaves with respect to changing triangulations. Let S' be a weak triangulation of X that contains S. Let $x \in X$. Let L be a complete valued extension of K such that X_L contains an L-rational point $\tilde{x}$ over x. Inside X_L, the disc $D(\tilde{x},S'_L)$ is included in $D(\tilde{x},S_L) \simeq D^-_L(0,R)$. Let R' be its radius as a sub-disc of $D^-_L(0,R)$ and set $\rho_{S',S}(x) = R'/R \in (0,1]$. Remark that it is constant and equal to 1 on Γ_S. It is now easy to check that, for any $i \in [\![1,\mathrm{rk}(\mathscr{F}_x)]\!]$, we have

$$\mathcal{R}_{S',i}(x,(\mathscr{F},\nabla)) = \min\left(\frac{\mathcal{R}_{S,i}(x,(\mathscr{F},\nabla))}{\rho_{S',S}(x)},1\right). \qquad (2.2.1)$$

2.3 Comparison with other definitions

We compare the radius of convergence we have introduced in definition 2.2.7 to other radii that appear in the literature.

2.3.1 *F. Baldassarri's definition using semistable models* The first definition of radius of convergence on a curve has been given by F. Baldassarri in [Bal10]. It was our main source of inspiration and our definition is very close to his.

Assume that the absolute value of K is non-trivial and that the curve X is strictly K-affinoid and compact. In this case, it is known that there exits a finite separable extension L/K such that the curve X_L admits a semistable formal model $\mathfrak{X}$ over L°.

There actually exists a strong relation between the semistable models of X_L and its triangulations (see [Duc, section 5.4], for a detailed account). Indeed, for any generic point $\tilde{\xi}$ of the special fibre $\mathfrak{X}_s$ of $\mathfrak{X}$, let ξ be the unique point of the generic fibre $\mathfrak{X}_\eta = X_L$ whose reduction is equal to $\tilde{\xi}$. Gathering all points ξ, we construct a finite set $S(\mathfrak{X})$, which is a triangulation of X_L. Let us remark that our notation unfortunately disagrees with F. Baldassarri's: in [Bal10], $S(\mathfrak{X})$ denotes the skeleton (*i.e.* $\Gamma_{S(\mathfrak{X})}$ using our notations) and not the triangulation.

Let $x \in X_L(L)$. In [Bal10], F. Baldassarri considers the biggest disc which does not meet the skeleton $\Gamma_{S(\mathfrak{X})}$ of $\mathfrak{X}$ (see definition 1.6.6). It is nothing but our disc $D(x,S(\mathfrak{X}))$. Since every point of $S(\mathfrak{X})$ has type 2, this disc is actually isomorphic to the open unit disc $D^-_L(0,1)$, and F. Baldassarri defines the radius of convergence $\mathcal{R}_{\mathfrak{X}}(x,\pi_L^*(\mathscr{F},\nabla))$ of $\pi_L^*(\mathscr{F},\nabla)$ at x as the absolute radius r of the biggest subdisc $D^-_L(0,r) \subset D^-_L(0,1)$ on which $(\mathscr{F},\nabla)$ is trivial (see [Bal10, definition 3.1.8]). This is compatible with our definition that uses a relative radius (see lemma 2.2.6 and remark 2.2.8). Finally, we have proved that, for any $x \in X_L(L)$, we have

$$\mathcal{R}_{\mathfrak{X}}(x,\pi_L^*(\mathscr{F},\nabla)) = \mathcal{R}_{S_L,1}(x,\pi_L^*(\mathscr{F},\nabla)).$$

F. Baldassarri extends his definition to other points of the curve by extending the scalars so as to make them rational (see [Bal10, definition 3.1.11]). One may check that, for any complete valued extension M of L, $\mathfrak{X}\hat{\otimes}_{L^\circ}M^\circ$ is as semistable model of X_M and that $S(\mathfrak{X}\hat{\otimes}_{L^\circ}M^\circ) = S(\mathfrak{X})_M$. Hence our definition coincides with F. Baldassarri's everywhere.

Let us also point out that, conversely, for any triangulation S of X that only contains points of type 2, there exits a finite separable extension L/K and a semistable formal model $\mathfrak{X}$ of X_L such that $S(\mathfrak{X}) = S_L$. Hence, under the hypotheses of this section and if we restrict to triangulations that contains only points of type 2, our definition is essentially equivalent to F. Baldassarri's.

Finally, let us mention that F. Baldassarri actually considers a slightly more general situation: $X = \bar{X} \setminus \{z_1,\dots,z_r\}$, where $\bar{X}$ is a compact curve as above and $z_1,\dots,z_r$ are K-rational points. In this case, he constructs the skeleton of X by branching on the skeleton of $\bar{X}$ a half-line ℓ_i that goes in the direction of z_i, for each i. The definition of radius of convergence may then be adapted.

Let us mention that this more general situation is already covered in our setting, since we did not require the curves to be compact. To find the same skeleton, it is enough to begin with the triangulation of $\bar{X}$ and add, for each i, a sequence of points that lies on ℓ_i and tend to z_i.

2.3.2 *The definition for analytic domains of the affine line* Assume that X is an analytic domain of the affine line $\mathbf{A}^{1,\mathrm{an}}_K$. The choice of a coordinate t on $\mathbf{A}^{1,\mathrm{an}}_K$ provides a global coordinate on X and it seems natural to use it in order to measure the radii of convergence. This normalisation has been used by F. Baldassarri and L. Di Vizio in [BDV08] (for the first radius) and by the second author in [Pul12]. We will call "embedded" the radii we define in this setting.

Let us first give a definition that does not refer to any triangulation.

Definition 2.3.1. Let x be a point of X and L be a complete valued extension of K such that X_L contains an L-rational point $\tilde{x}$ over x. Let $D(\tilde{x},X_L)$ be the biggest open disc centred at $\tilde{x}$ that is contained in X_L.

Let us consider the pull-back $(\tilde{\mathscr{F}},\tilde{\nabla})$ of $(\mathscr{F},\nabla)$ on $D(\tilde{x},X_L)$. Let $r = \mathrm{rk}(\mathscr{F}_x)$. For $i \in [\![1,r]\!]$, we denote $\mathcal{R}^{\mathrm{emb}}_i(x,(\mathscr{F},\nabla))$ the radius of the biggest open disc centred at $\tilde{x}$, measured using the coordinate t on $\mathbf{A}^{1,\mathrm{an}}_L$, on which the connection $(\tilde{\mathscr{F}},\tilde{\nabla})$ admits at least $r-i+1$ horizontal sections which are linearly independent over L.

As before, one checks that the definition of $\mathcal{R}^{\mathrm{emb}}_i(x,(\mathscr{F},\nabla))$ only depends on the point x and not on L or $\tilde{x}$. This radius is denoted $\mathcal{R}^M_i(x)$ in [Pul12, section 4.3]. There, he actually works over an affinoid domain V of the affine line. Over V, the sheaf $\mathscr{F}$ is free and, if $\mathscr{O}(V)$ is endowed with the usual derivation $d = \mathrm{d}/\mathrm{d}t$, the pair $(\mathscr{F},\nabla)$ corresponds to a differential module (M,D) over $(\mathscr{O}(V),d)$. This is where the M in the notation comes from.

Although possibly superfluous in this context, it is also possible to state a definition that depends on a weak triangulation S of X.

Definition 2.3.2. Let x be a point of X and L be a complete valued extension of K such that X_L contains an L-rational point $\tilde{x}$ over x. As in definition 2.2.1, consider $D(\tilde{x},S_L)$, the biggest open disc centred at $\tilde{x}$ that is contained in $X_L \setminus S_L$. We denote $\rho_S(x)$ its radius, measured using the coordinate t on $\mathbf{A}_L^{1,\mathrm{an}}$.

Let us consider the pull-back $(\tilde{\mathscr{F}}, \tilde{\nabla})$ of $(\mathscr{F}, \nabla)$ on $D(\tilde{x}, S_L)$. Let $r = \mathrm{rk}(\mathscr{F}_x)$. For $i \in [\![1, r]\!]$, we denote $\mathcal{R}_{S,i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))$ the radius (possibly $+\infty$) of the biggest open disc centred at $\tilde{x}$, measured using the coordinate t on $\mathbf{A}_L^{1,\mathrm{an}}$, on which the connection $(\tilde{\mathscr{F}}, \tilde{\nabla})$ admits at least $r-i+1$ horizontal sections which are linearly independent over L.

Once again, the definitions of $\rho_S(x)$ and $\mathcal{R}_{S,i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))$ are independent of the choices of L and $\tilde{x}$. This radius is denoted $\mathcal{R}_{S,i}^{M}(x)$ in [Pul12, section 8].

The radii we have just defined may easily be linked to the one we introduced in definition 2.2.7. The simplest case is the second: for any $i \in [\![1, \mathrm{rk}(\mathscr{F}_x)]\!]$, we have

$$\mathcal{R}_{S,i}(x, (\mathscr{F}, \nabla)) = \frac{\mathcal{R}_{S,i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))}{\rho_S(x)}. \tag{2.3.1}$$

Let us now remark that any analytic domain of the affine line $\mathbf{A}_K^{1,\mathrm{an}}$ admits a smallest triangulation. Assume that X is not the affine line itself and let S_0 be its smallest triangulation. For any $i \in [\![1, \mathrm{rk}(\mathscr{F}_x)]\!]$, we have

$$\mathcal{R}_{S_0,i}(x, (\mathscr{F}, \nabla)) = \frac{\mathcal{R}_{S_0,i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))}{\rho_{S_0}(x)} = \frac{\mathcal{R}_{i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))}{\rho_{S_0}(x)}. \tag{2.3.2}$$

3. The result

We have just defined the radius of convergence of $(\mathscr{F}, \nabla)$ at any point of the curve X. We now investigate its properties.

3.1 Statement

Let us state precisely the result we are interested in. We will first define the notion of locally finite subgraph of X.

From the existence of weak triangulations, one deduces that every point of X has a neighbourhood which is uniquely arcwise connected. In particular, the curve X may be covered by uniquely arcwise connected analytic domains. On such a subset, it makes sense to speak of the segment $[x, y]$ joining two given points x and y, hence of convex subsets (see also [BR10, section 2.5]).

Definition 3.1.1. A subset Γ of X is said to be a (resp. locally) finite subgraph of X if there exists a (resp. locally) finite family $\mathscr{V}$ of affinoid domains of X that covers Γ and such that, for every element V of $\mathscr{V}$, we have

i) V is uniquely arcwise connected;

ii) $\Gamma \cap V$ is the convex hull of a finite number of points.

Theorem 3.1.2. *Under the assumptions of setting 1.2, the map*

$$x \in X \mapsto \boldsymbol{\mathcal{R}}_S(x, (\mathscr{F}, \nabla))$$

is continuous on X and locally constant outside a locally finite subgraph Γ of X.

Remark 3.1.3. Let us enlarge Γ to a locally finite subgraph Γ' of X such that *i)* Γ' contains Γ_S ;

ii) Γ' meets every connected component of X ;

iii) for any connected component V of $X \setminus S$, the graph $\Gamma' \cap V$ is convex.

In this case, there is a natural continuous retraction $X \to \Gamma'$ and the map $\boldsymbol{\mathcal{R}}_S(\cdot, (\mathscr{F}, \nabla))$ factorizes by it.

Remark 3.1.4. In the case of analytic domains of the affine line, the map $\boldsymbol{\mathcal{R}}_S(\cdot, (\mathscr{F}, \nabla))$ satisfies additional properties (see [Pul12, theorem 4.7]). Since, by definition of a weak triangulation, $X \setminus S$ is a union of virtual open discs and annuli, some of them may be transferred to the case of curves. For instance, we may assume that the set of vertices of the graph Γ contains the weak triangulation S. In this case, every open edge e of Γ may be embedded into the affine line. Let $r = \mathrm{rk}(\mathscr{F}_{|e})$. We deduce that, for any $i \in [\![1, r]\!]$, the restriction of $\mathcal{R}_{S,i}(\cdot, (\mathscr{F}, \nabla))$ to e is piecewise log-linear, with slopes of the form m/n, with $m \in \mathbf{Z}$ and $n \in [\![1, r]\!]$.

The continuity of the radius of convergence $\mathcal{R}_{S,1}$ has been proven by F. Baldassarri and L. Di Vizio in the case of affinoid domains of the affine line (see [BDV08]) and by F. Baldassarri in general (see [Bal10]). His setting is actually slightly less general than ours, but his result extends easily.

For the multiradius of convergence on affinoid domains of the affine line, the result is due to the second author (see [Pul12, theorem 4.7] for the case of the minimal weak triangulation and section 8 in general).

Theorem 3.1.5 (A. Pulita). *Assume that X is an affinoid domain of the affine line. Then the map $\boldsymbol{\mathcal{R}}_S(\cdot, (\mathscr{F}, \nabla))$ is continuous and factorizes by the retraction through a finite subgraph of X containing S. In particular, it is locally constant outside this graph.*

Writing open discs (resp. annuli) as increasing unions of closed discs (resp. annuli), we deduce the following result.

Corollary 3.1.6. *Assume that X is an open disc, an open annulus or a semi-open annulus. Then the map $\boldsymbol{\mathcal{R}}_S(\cdot, (\mathscr{F}, \nabla))$ is continuous and factorizes by the retraction through a locally finite subgraph of X. In particular, it is locally constant outside this graph.*

By lemma 2.2.9, it is enough to prove the result after a scalar extension. *From now on, we will assume that K is algebraically closed.*

3.2 A geometric result

The overall strategy of our proof is the following. By definition of a weak triangulation, $X \setminus S$ is a union of open discs and annuli and, for those, we may use the results of the second author (see [Pul12, theorem 4.7]). We still need to investigate what happens around the points of the triangulation S. To carry out this task, we will write the curve X, locally around those points, as a finite étale cover of a subset of the affine line, consider the push-forward of $(\mathscr{F}, \nabla)$, which is well understood thanks to [Pul12, theorem 4.7] again, and relate its radii of convergence to the original radii.

We will need to find étale morphisms from open subsets of X to the affine line that satisfy nice properties. The main result we use has been adapted from A. Ducros's manuscript (see the proof of theorem 3.4.1). For the convenience of the reader, we have decided to sketch the proof. Let us add a word of caution. Here, and everywhere in this article, we use A. Ducros's notion of "branch": it roughly corresponds to a direction out of a point (see [Duc, section 1.7] for a precise definition) and should not be confused with A. Pulita's notion (which will be referred to here as interval or segment inside the curve). The image of a branch by a morphism of curves is a branch and the preimage of a branch is a finite union of branches. A section of a branch out of a point x is a connected open subset U of X belonging to the branch such that x belongs to the closure $\bar{U}$ of U but not to U itself.

Let x be a point of X of type 2 and consider the complete valued field $\mathscr{H}(x)$ associated to it. Its residue field $\widetilde{\mathscr{H}(x)}$ is the function field of a projective normal and integral algebraic curve $\mathscr{C}$ over $\tilde{k}$, called the residual curve (see [Duc, 2.2.3.1]). If x lies in the interior of X,the closed points of $\mathscr{C}$ correspond bijectively to the branches out of the point x (see [Duc, 3.2.11.1]).

We say that a property holds for almost every element of a set E if it holds for every element of E except a finite number of them.

Theorem 3.2.1 (A. Ducros). *Let x be a point of X of type 2. There exists an affinoid neighbourhood Y of X containing x, an affinoid domain W of $\mathbf{P}^{1,an}_K$ and a finite étale map $\psi : Y \to W$ such that*

i) $\psi^{-1}(\psi(x)) = \{x\}$*;*

ii) almost every connected component of $Y \setminus \{x\}$ is an open unit disc with boundary $\{x\}$;

iii) almost every connected component of $W \setminus \{\psi(x)\}$ is an open unit disc with boundary $\{\psi(x)\}$;

iv) for almost every connected component V of $Y \setminus \{x\}$, the induced morphism $V \to \psi(V)$ is an isomorphism.

Let b be a branch out of x. Moreover, the map ψ may be constructed so as to satisfy any of the following properties:

a) The morphism ψ induces an isomorphism between a section of b and a section of $\psi(b)$.

b) The degree d of ψ is prime to p and $\psi^{-1}(\psi(b)) = \{b\}$.

Proof. Let us first assume that X is the analytification $\mathscr{X}^{\mathrm{an}}$ of a smooth connected projective algebraic curve $\mathscr{X}$. Let $\mathscr{C}$ be the residual curve at the point x. Let g be a rational function on $\mathscr{C}$ that induces a generically étale morphism $\mathscr{C} \to \mathbf{P}^1_{\tilde{k}}$. Let f be a rational function on $\mathscr{X}$ such that $|f(x)| = 1$ and $\widetilde{f(x)} = g$. Let $f^{\mathrm{an}} : \mathscr{X}^{\mathrm{an}} \to \mathbf{P}^{1,\mathrm{an}}_k$ be the associated morphism.

Let us first remark that, for almost every connected component V of $\mathscr{X}^{\mathrm{an}} \setminus \{x\}$, V meets a unique branch out of x and $f^{\mathrm{an}}(V)$ is a connected component of $\mathbf{P}^{1,\mathrm{an}}_K \setminus \{f^{\mathrm{an}}(x)\}$.

Since the map induced by g is generically étale, for almost every connected component V of $\mathscr{X}^{\mathrm{an}} \setminus \{x\}$, it is unramified at the closed point of $\mathscr{C}$ corresponding to the branch associated to V. From this we deduce that the morphism $V \to f^{\mathrm{an}}(V)$ induced by f^{an} has degree 1 (see [Duc, théorème 3.3.16]), hence is an isomorphism.

Finally, we choose an affinoid neighbourhood W of $f^{\mathrm{an}}(x)$ in $\mathbf{P}^{1,\mathrm{an}}_K$ such that the different points of $(f^{\mathrm{an}})^{-1}(f^{\mathrm{an}}(x))$ belong to different connected components of $(f^{\mathrm{an}})^{-1}(W)$. Let Y be the connected component containing x and $\psi : Y \to W$ be the induced morphism. Since $\psi(x)$ is a point of type 2, property *iii)* is clear. The other three are satisfied by construction.

The case of a general quasi-smooth curve X reduces to this one since there exists an affinoid neighbourhood of X containing x that may be embedded in the analytification of a smooth projective algebraic curve $\mathscr{X}$.

Let us now consider the additional properties. Let z be the closed point of the residual curve $\mathscr{C}$ corresponding to the branch b.

Let us begin with property a. Using the theorem of Riemann-Roch, one proves that there exists a rational function g on $\mathscr{C}$ with a single zero at z. The morphism f^{an} constructed as above will have degree 1 over the branch b, hence induce an isomorphism on a suitable section of b.

As regards property b, it may be deduced from a result of O. Gabber (see [Duc, proof of lemme 3.3.2]): there exists a integer d which is prime to p and a rational function g on $\mathscr{C}$ of degree d that has a zero of order d in z. □

3.3 Proof of the finiteness property

In this section, we will prove that the map $\boldsymbol{\mathcal{R}}_S(\cdot,(\mathscr{F},\nabla))$ is locally constant outside a locally finite subgraph Γ of X.

By definition of a triangulation, $X \setminus S$ is a union of open discs and annuli, each of which may be handled by theorem 3.1.5. We still need to investigate the behaviour of the multiradius around the points of the triangulation. Let us first remark that, as far as the finiteness property is concerned, it is harmless to change triangulations.

Lemma 3.3.1. *Let S and S' be two weak triangulations of X. There exists a locally finite subgraph Γ of X outside which the map $\boldsymbol{\mathcal{R}}_S(\cdot,(\mathscr{F},\nabla))$ is locally constant if, and only if, there exists a locally finite subgraph Γ' of X outside which the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,(\mathscr{F},\nabla))$ is locally constant.*

Proof. It is possible to construct a triangulation S'' that contains both S and S'. Hence we may assume that $S \subset S'$. In this case, it is clear that the property for S implies the property for S'.

Let us assume that there exists a locally finite subgraph Γ' of X outside which the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,(\mathscr{F},\nabla))$ is locally constant. We may assume that Γ' contains $\Gamma_{S'}$. Let U be a connected component of $X \setminus \Gamma'$. It is enough to prove that the map $\boldsymbol{\mathcal{R}}_S(\cdot,(\mathscr{F},\nabla))$ is constant on U. Let V be the connected component of $X \setminus \Gamma_S$ that contains U. Both U and V are discs and the distance function $\rho_{S,S'}$ (see the paragraph preceding formula (2.2.1) is constant on U. Let ρ be its value. Let $r = \mathrm{rk}(\mathscr{F}_{|U})$. For any $i \in [\![1,r]\!]$ and any $x \in U$, we have

$$\mathcal{R}_{S',i}(x,(\mathscr{F},\nabla)) = \min\left(\frac{\mathcal{R}_{S,i}(x,(\mathscr{F},\nabla))}{\rho}, 1\right).$$

We now have two cases. Fix $i \in [\![1,r]\!]$. If there exists $x \in U$ such that $\mathcal{R}_{S,i}(x,(\mathscr{F},\nabla))$ is at least ρ, then $\mathcal{R}_{S,i}(\cdot,(\mathscr{F},\nabla))$ is constant on U (which is contained in an open disc of relative radius ρ). Otherwise, the maps $\mathcal{R}_{S,i}(\cdot,(\mathscr{F},\nabla))$ and $\rho\,\mathcal{R}_{S',i}(\cdot,(\mathscr{F},\nabla))$ coincide on U, hence both are constant. □

In our study, we will need to restrict the connection to some subspaces. Unfortunately, the multiradius of convergence may vary under this operation. In the following lemma, we gather a few easy cases where the resulting multiradius may be controlled.

Notation 3.3.2. For any $R > 0$, we denote by η_R the unique point of the Shilov boundary of the disc $D^+(0,R)$.

Lemma 3.3.3. *Let x be a point of S. Let C be an open disc or annulus inside X such that $\bar{C} \cap S = \{x\}$.*

a) Assume that C is an open disc. Then, for any $y \in C$, we have

$$\boldsymbol{\mathcal{R}}_\emptyset(y,(\mathscr{F},\nabla)_{|C}) = \boldsymbol{\mathcal{R}}_S(y,(\mathscr{F},\nabla)).$$

b) Assume that C is an open annulus such that $\Gamma_C \cap \Gamma_S \neq \emptyset$. Then, for any $y \in C$, we have

$$\boldsymbol{\mathcal{R}}_{\emptyset}(y, (\mathscr{F}, \nabla)_{|C}) = \boldsymbol{\mathcal{R}}_S(y, (\mathscr{F}, \nabla)).$$

c) Assume that C is an open annulus such that $\Gamma_C \cap \Gamma_S = \emptyset$. Identify C with an annulus $C^-(0; R_1, R_2)$, with coordinate t, in such a way that $\lim_{R \to R_2} \eta_R = x$. Then, for any $y \in C$ and any $i \in [\![1, \mathrm{rk}(\mathscr{F}_{|C})]\!]$, we have

$$\mathcal{R}_{\emptyset,i}(y, (\mathscr{F}, \nabla)_{|C}) = \min\left(\frac{R_2}{|t(y)|}\, \mathcal{R}_{S,i}(y, (\mathscr{F}, \nabla)), 1\right).$$

Proof. Assume we are in case a. The open subset C is isomorphic to an open disc and the point x lies at its boundary. As a consequence, for any complete valued extension L of K and any L-rational point $\tilde{y}$ of X_L, the disc $D(\tilde{y}, S_L)$ is equal to C_L. In particular, the multiradius of convergence of $(\mathscr{F}, \nabla)$ on C only depends on the restriction of $(\mathscr{F}, \nabla)$ to C, and the result follows.

Assume we are in case b. We actually have $\Gamma_C = C \cap \Gamma_S$ and the result is proved as above.

Assume we are in case c, which means that there is a disc D inside X that contains C. When one restricts the connection to C, one creates a new branch of the skeleton, which amounts to adding the point η_{R_1} to the triangulation S. The result now follows from case a and formula (2.2.1). □

Using theorem 3.2.1, we now prove a kind of generic finiteness of the multiradius around a point of the triangulation which is of type 2.

Proposition 3.3.4. *Let x be a point of S of type 2. There exists an affinoid domain Y_x of X such that $Y_x \cap S = \{x\}$ and, for every $i \in [\![1, r]\!]$, the map $\mathcal{R}_{S,i}(\cdot, (\mathscr{F}, \nabla))$ is locally constant outside a finite subgraph Γ_x of Y_x that contains x.*

Proof. Let us consider a finite étale map $\psi : Y_x \to W_x$ as in theorem 3.2.1. It is possible to restrict W_x and Y_x by removing a finite number of branches in order to assume that $S \cap Y_x = \{x\}$ and that the conditions *ii)*, *iii)* and *iv)* hold for every connected component that appears in their statements. Beware that Y_x will no longer be a neighbourhood of x.

We may consider the push-forward $\psi_*(\mathscr{F}, \nabla)$ of $(\mathscr{F}, \nabla)$ to W_x. The subset $T = \{\psi(x)\}$ of W_x is a weak triangulation of W_x.

Let V be a connected component of $Y_x \setminus \{x\}$. Let d be the degree of ψ. The preimage $V' = \psi^{-1}(\psi(V))$ of $\psi(V)$ consists of d connected components $V_1 = V, V_2, \ldots, V_d$, each of them isomorphic to an open disc. For each of them, the induced morphism $\psi_i : V_i \to \psi(V)$ is an isomorphism. By case a of lemma 3.3.3, we have

$$\forall y \in V', \boldsymbol{\mathcal{R}}_S(y, (\mathscr{F}, \nabla)) = \boldsymbol{\mathcal{R}}_{\emptyset}(y, (\mathscr{F}, \nabla)_{|V'})$$

and

$$\begin{aligned}\forall z \in \psi(V), \boldsymbol{\mathcal{R}}_T(z, \psi_*(\mathscr{F}, \nabla)) &= \boldsymbol{\mathcal{R}}_{\emptyset}(z, \psi_*(\mathscr{F}, \nabla)_{|\psi(V)}) \\ &= \boldsymbol{\mathcal{R}}_{\emptyset}(z, \psi'_*((\mathscr{F}, \nabla)_{|V'})),\end{aligned}$$

where we denote $\psi' : V' \to \psi(V)$ the induced morphism.

Since ψ' is a trivial cover of degree d, the situation is simple. Actually, the module $\psi'_*((\mathscr{F}, \nabla)_{|V'})$ over $\psi(V)$ splits as $\bigoplus_{1 \leqslant i \leqslant d} \psi_{i*}((\mathscr{F}, \nabla)_{|V_i})$. By [Pul12, proposition 5.5], we deduce that, for every $y \in V$, every component of the multiradius of convergence $\boldsymbol{\mathcal{R}}_S(y, (\mathscr{F}, \nabla))$ appears in $\boldsymbol{\mathcal{R}}_T(\psi(y), \psi_*(\mathscr{F}, \nabla))$.

By theorem 3.1.5, the map $\boldsymbol{\mathcal{R}}_T(\cdot, \psi_*(\mathscr{F}, \nabla))$ is locally constant outside a finite subgraph Γ_x of W_x. We may assume that Γ_x contains x. Let U be a connected open subset of $Y_x \setminus \psi^{-1}(\Gamma_x)$. It is contained in an open disc or annulus, hence the map $\boldsymbol{\mathcal{R}}_S(\cdot, (\mathscr{F}, \nabla))$ is continuous on U by corollary 3.1.6. For every point $y \in U$, the components of the r-tuple $\boldsymbol{\mathcal{R}}_S(y, (\mathscr{F}, \nabla))$ are equal to some of the components of the dr-tuple $\boldsymbol{\mathcal{R}}_T(\psi(y), \psi_*(\mathscr{F}, \nabla))$. Since $\boldsymbol{\mathcal{R}}_T(\cdot, \psi_*(\mathscr{F}, \nabla))$ is constant on $\psi(U)$, there are only finitely many possible such values. We deduce that the restriction of the map $\boldsymbol{\mathcal{R}}_S(\cdot, (\mathscr{F}, \nabla))$ to U is continuous with values in a finite set, hence constant. □

Let x be a point of S of type 2. The affinoid domain Y_x of the previous corollary contains entirely all the branches out of x except a finite number of them. We still need to prove the finiteness property on the remaining branches. This is the object of the following proposition (where we also handle points of type 3).

Proposition 3.3.5. *Let x be a point of S and b be a branch out of it. Let C be an open disc or annulus which is a section of b. There exists a sub-annulus $C_{x,b}$ of C which is a section of b that satisfies $\bar{C}_{x,b} \cap S = \{x\}$ and a finite subgraph $\Gamma_{x,b}$ of $\bar{C}_{x,b}$ such that the map $\boldsymbol{\mathcal{R}}_\emptyset(\cdot, (\mathscr{F}, \nabla)_{|C_{x,b}})$ is locally constant outside $\Gamma_{x,b} \cap C_{x,b}$.*

Proof. By [Duc, théorème 3.3.5], any point of type 3 has a neighbourhood which is isomorphic to a closed annulus. Hence, around such a point, we may conclude by theorem 3.1.5.

Let us now assume that x is a point of type 2. The proof will closely follow that of proposition 3.3.4. Let us consider a finite étale map $\psi : Y \to W$ as in theorem 3.2.1 that satisfies the additional condition a. Let $C_{x,b}$ be an open sub-annulus of C which is a section of b that satisfies $\bar{C}_{x,b} \cap S = \{x\}$ and such that the induced map $\psi_0 : C_{x,b} \to \psi(C_{x,b})$ is an isomorphism. We may assume that $C_{x,b}$ is not a disc. Let us consider the push-forward $\psi_*(\mathscr{F}, \nabla)$ of the connection $(\mathscr{F}, \nabla)$ to W. We endow W with a weak triangulation T such that $T \cap \psi(\bar{C}_{x,b}) = \partial\psi(C_{x,b}) = \psi(\partial C_{x,b})$.

Arguing as in the proof of proposition 3.3.4, we show that

$$\forall z \in \psi(C_{x,b}), \boldsymbol{\mathcal{R}}_T(z, \psi_*(\mathscr{F}, \nabla)) = \boldsymbol{\mathcal{R}}_\emptyset(z, \psi'_*((\mathscr{F}, \nabla)_{|P_{x,b}})),$$

where we denote $\psi' : P_{x,b} = \psi^{-1}(\psi(C_{x,b})) \to \psi(C_{x,b})$ the induced morphism.

The subset $P_{x,b}$ has several connected components, one of which is $C_{x,b}$. We deduce that $(\psi_0)_*((\mathscr{F}, \nabla)_{|C_{x,b}})$ is a direct factor of $\psi'_*((\mathscr{F}, \nabla)_{|P_{x,b}})$. Since ψ_0 is an isomorphism, for every $y \in C_{x,b}$, every component of the multiradius $\boldsymbol{\mathcal{R}}_\emptyset(y, (\mathscr{F}, \nabla)_{|C_{x,b}})$ appears in $\boldsymbol{\mathcal{R}}_T(\psi(y), \psi_*(\mathscr{F}, \nabla))$. By theorem 3.1.5, the map $\boldsymbol{\mathcal{R}}_T(\cdot, \psi_*(\mathscr{F}, \nabla))$ is locally constant outside a finite subgraph Γ of W. Using an argument of continuity as in the last paragraph of the proof of proposition 3.3.4, we deduce that the map $\boldsymbol{\mathcal{R}}_\emptyset(\cdot, (\mathscr{F}, \nabla)_{|C_{x,b}})$ is locally constant outside $\psi^{-1}(\Gamma)$. Remark that $\psi^{-1}(\Gamma)$ is a finite subgraph of $\bar{C}_{x,b}$. □

Remark 3.3.6. In the situation of proposition 3.3.5, the coefficients of the matrix of the connection on C converge in a neighbourhood of $\bar{C}$. If X were an affinoid domain of the affine line, we would deduce that these coefficients are analytic elements on C and then conclude by [Pul12, corollary 4.8]. Unfortunately, in the general case, such functions do not give rise to analytic elements.

We may now conclude the proof of the finiteness property. We change the triangulation S of X into a triangulation S' such that every annulus $C_{x,b}$ of proposition 3.3.5 is a connected component of $X \setminus S'$. (We may have to restrict the annuli $C_{x,b}$ to do so, which does not lead to any trouble.) By lemma 3.3.1, this does not affect the result we want to prove.

Let us have a look at the connected components of $X \setminus S'$. For those that belong to some affinoid domain Y_x as in proposition 3.3.4, we constructed a finite graph Γ_x out of which the map $\boldsymbol{\mathcal{R}}_S(\cdot,(\mathscr{F},\nabla))$, hence also the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,(\mathscr{F},\nabla))$ is locally constant. The connected components of the form $C_{x,b}$ are dealt with in proposition 3.3.5. For those, we constructed a finite subgraph $\Gamma_{x,b}$ of X such that the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,(\mathscr{F},\nabla)_{|C_{x,b}})$ (by case b of lemma 3.3.3) is locally constant outside $\Gamma_{x,b}\cap C_{x,b}$.

The connected components of $X \setminus S'$ that remain are discs or annuli that lie away from the triangulation. For instance, if such a connected component is an open disc D, there exists a closed disc $D^+(0,R)$, with $R>0$, inside X such that $\eta_R \in S'$ and $D = D^-(0,R)$. Using theorem 3.1.5 (and case a of lemma 3.3.3), we deduce that there exists a finite subgraph Γ_D of $D^+(0,R)$ such that the restriction of the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,(\mathscr{F},\nabla))$ to D is locally constant outside $\Gamma_D\cap D$. The case of an annulus C is handled the same way (using theorem 3.1.5 if C has two boundary points in X, corollary 3.1.6 otherwise and case b of lemma 3.3.3). This concludes the proof of the finiteness property.

3.4 Proof of the continuity property

To finish the proof of theorem 3.1.2, we need to prove that the multiradius of convergence $\boldsymbol{\mathcal{R}}_S(\cdot,(\mathscr{F},\nabla))$ is continuous on X. By definition of a weak triangulation, $X\setminus S$ is a disjoint union of open discs and annuli. Hence, the continuity on $X\setminus S$ follows readily from corollary 3.1.6. We will prove the continuity at points of the triangulation S.

Let $x\in S$. If x has type 3, by [Duc, théorème 3.3.5], it has a neighbourhood which is isomorphic to an annulus and we are done. Let us now assume that x has type 2. Thanks to the finiteness property, it is actually enough to prove the continuity at x of $\boldsymbol{\mathcal{R}}_S(\cdot,(\mathscr{F},\nabla))$ on any segment of the form $[y,x]$ inside X. We may assume that the interval $[y,x)$ lies inside the skeleton of an open annulus C whose closure contains x.

Before proving the continuity, let us state a result that relates the multiradius of convergence after push-forward by an étale map to the original multiradius of convergence. We will only be interested in points that lie on the skeleton of the triangulation.

Lemma 3.4.1. *Let Z be a quasi-smooth K-analytic curve endowed with a weak triangulation T. Let $\psi : X \to Z$ be a finite étale morphism. Let $x\in\Gamma_S\cap\psi^{-1}(\Gamma_T)$. Assume that $d=[\mathscr{H}(x):\mathscr{H}(\psi(x))]$ is prime to p*[3]*. Then, for every $i\in[\![1,\mathrm{rk}(\mathscr{F}_x)]\!]$ and $j\in[\![1,d]\!]$, we have*

$$\mathcal{R}_{T,d(i-1)+j}(\psi(x),\psi_*(\mathscr{F},\nabla)) = \mathcal{R}_{S,i}(x,(\mathscr{F},\nabla)).$$

Proof. Since $x\in\Gamma_S$ and $\psi(x)\in\Gamma_T$, the radii of convergence at those points only depend on the restrictions of the connections to those points (see remark 2.2.2). Hence we may localize and assume that $\psi^{-1}(\psi(x))=\{x\}$ and that ψ has degree d. Let L be an algebraically closed complete valued extension of $\mathscr{H}(x)$.

By definition of x_L (see definition 2.1.5), the norm associated to x_L is induced by the tensor norm on $\mathscr{H}(x)\hat{\otimes}_K L$. Hence

$$\sqrt{|\mathscr{H}(x_L)^*|} = \sqrt{\|\mathscr{H}(x)\hat{\otimes}_K L\setminus\{0\}\|} = \sqrt{|L^*|},$$

since L contains $\mathscr{H}(x)$. We deduce that the point x_L has type 2. As a consequence, the point $\psi_L(x_L)$, where $\psi_L : X_L\to Z_L$ is the morphism induced by ψ, also has type 2.

[3] If $\tilde{K}$ has characteristic 0, then $p=1$ and this condition is always satisfied.

Moreover, we have an isometric embedding $\mathscr{H}(x)\hat{\otimes}_K L \hookrightarrow \mathscr{H}(x_L)$, hence an isometric embedding $\mathscr{H}(\psi(x))\hat{\otimes}_K L \hookrightarrow \mathscr{H}(x_L)$. We deduce that

$$\psi_L(x_L) = \psi(x)_L.$$

Let $\mathscr{C}$ and $\mathscr{D}$ be the residual curves at x_L and $\psi(x)_L$ respectively. The morphism $\mathscr{C} \to \mathscr{D}$ induced by ψ has degree prime to p, hence it is generically étale, since $\tilde{K}$ is algebraically closed. Arguing as in the proof of theorem 3.2.1, we deduce that almost every connected component of $\pi_L^{-1}(\psi(x)) \setminus \{\psi(x)_L\}$ is a disc over which the morphism ψ_L induces a trivial cover of degree d.

Let W be such a connected component and $\tilde{y}$ be an L-rational point of it. Let us denote $\psi_L^{-1}(\tilde{y}) = \{\tilde{x}_1, \dots, \tilde{x}_d\}$. Arguing as in the proof of proposition 3.3.4, we prove that $\boldsymbol{\mathcal{R}}_{T_L}(\tilde{y}, \psi_{L*}\pi_L^*(\mathscr{F}, \nabla))$ is obtained by concatenating and reordering the tuples $\boldsymbol{\mathcal{R}}_{S_L}(\tilde{x}_1, \pi_L^*(\mathscr{F}, \nabla)), \dots, \boldsymbol{\mathcal{R}}_{S_L}(\tilde{x}_d, \pi_L^*(\mathscr{F}, \nabla))$.

We have $\pi_L(\tilde{y}) = \psi(x)$ and, for any $i \in [\![1, d]\!]$, $\pi_L(\tilde{x}_i) = x$. From lemma 2.2.9, we deduce that $\boldsymbol{\mathcal{R}}_{T_L}(\tilde{y}, \psi_{L*}\pi_L^*(\mathscr{F}, \nabla)) = \boldsymbol{\mathcal{R}}_T(\psi(x), \psi_*(\mathscr{F}, \nabla))$ and, for any $i \in [\![1, d]\!]$, $\boldsymbol{\mathcal{R}}_{S_L}(\tilde{x}_i, \pi_L^*(\mathscr{F}, \nabla)) = \boldsymbol{\mathcal{R}}_{S_L}(x, (\mathscr{F}, \nabla))$. The result follows. □

Proposition 3.4.2. *Let x be a point of S of type 2 and b be a branch out of x. Let C be an open annulus which is a section of b and y be a point in the skeleton Γ_C of C. For every $i \in [\![1, \mathrm{rk}(\mathscr{F}_x)]\!]$, the restriction of the map $\mathcal{R}_{S,i}(\cdot, (\mathscr{F}, \nabla))$ to the segment $[y, x]$ is continuous at the point x.*

Proof. Let us consider a finite étale map $\psi : Y \to W$ that satisfies the conclusions of theorem 3.2.1 and condition b. In particular, the degree d of ψ is prime to p. Replacing C by a sub-annulus, we may assume that $S \cap C = \emptyset$ and that, for any $z \in \Gamma_C$, $\psi^{-1}(\psi(z)) = \{z\}$ (hence $[\mathscr{H}(z) : \mathscr{H}(\psi(z))] = d$). Let us endow W with a triangulation T whose skeleton contains $\psi(\Gamma_C)$.

Let us first assume that $\Gamma_C \cap \Gamma_S \neq \emptyset$. By lemmas 3.3.3, case b, and 3.4.1, for any $z \in \Gamma_C$ and any $i \in [\![1, \mathrm{rk}(\mathscr{F}_x)]\!]$, we have

$$\mathcal{R}_{S,i}(z, (\mathscr{F}, \nabla)) = \mathcal{R}_{T,di}(\psi(z), \psi_*(\mathscr{F}, \nabla)).$$

Let us now assume that $\Gamma_C \cap \Gamma_S = \emptyset$. By case c of lemma 3.3.3, whose notations we borrow, for any $i \in [\![1, \mathrm{rk}(\mathscr{F}_x)]\!]$ and any $R \in (R_1, R_2)$, we have

$$\mathcal{R}_{T,di}(\psi(\eta_R), \psi_*(\mathscr{F}, \nabla)) = \min\left(\frac{R_2}{R}\, \mathcal{R}_{S,i}(\eta_R, (\mathscr{F}, \nabla)), 1\right).$$

By lemma 3.4.1 again, for any $i \in [\![1, \mathrm{rk}(\mathscr{F}_x)]\!]$, we have

$$\mathcal{R}_{S,i}(x, (\mathscr{F}, \nabla)) = \mathcal{R}_{T,di}(\psi(x), \psi_*(\mathscr{F}, \nabla)).$$

The result now follows from the continuity of $\mathcal{R}_{T,di}(\cdot, \psi_*(\mathscr{F}, \nabla))$ on W (see theorem 3.1.5). It is obvious in the first case and easy to prove in the second using the fact that radii are always at most 1. □

This concludes the proof of the continuity of the multiradius and the proof of theorem 3.1.2.

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