The convergence Newton polygon of a p-adic differential equation IV: local index theorems

Jérôme Poineau and Andrea Pulita

Contents

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Here and for the rest of the text, we fix an ultrametric complete valued field $(K, |\cdot|)$ of characteristic 0. We denote by $\widetilde{K}$ its residue field and by p be the characteristic exponent of the latter (either 1 or a prime number). We fix an algebraic closure K^{alg} of K. The absolute value $|\cdot|$ on K extends uniquely to it and we denote it identically. We denote by $(\widehat{K^{\mathrm{alg}}}, |\cdot|)$ its completion.

We also set $\omega := \liminf_n |n!|^{1/n}$ (the radius of convergence of the exponential series). One has

$$\omega \;=\; \begin{cases} 1 & \text{if the valuation of } K \text{ is trivial on } \mathbb{Q}, \\ |p|^{\frac{1}{p-1}} & \text{if the valuation of } K \text{ is } p\text{-adic on } \mathbb{Q}. \end{cases} \tag{0.1}$$

1. Definitions and notations

In the whole paper, we will use the definitions and notation from [PP13] and we refer the reader to this manuscript (and especially to its first section).

1.1. Radii of convergence

In this paper, we will mostly be interested in radii of convergence of differential equations over pseudo-annuli. We recall here the basic definitions.

Definition 1.1.1 (Virtual open disk)**.** *A non-empty connected K-analytic space is called a virtual open disk if it becomes isomorphic to a disjoint union of open disks over $\widehat{K^{\mathrm{alg}}}$.*

Following [Duc, 5.1.8], we may now define the analytic skeleton of an analytic curve.

Definition 1.1.2 (Analytic skeleton)**.** *We call* analytic skeleton *of an analytic curve X the set of points that have no neighborhoods isomorphic to a virtual open disk. We usually denote it by Γ_X.*

Note that the analytic skeleton Γ_D of a virtual open disk D is empty.

Definition 1.1.3 (Open pseudo-annulus)**.** *Assume that K is algebraically closed. We say that a connected quasi-smooth K-analytic curve C is an* open pseudo-annulus *if*

i) it has no boundary;

ii) it contains no points of positive genus;

iii) its analytic skeleton Γ_C is an open segment.

We call Γ_C the skeleton of C.

If K is arbitrary, we say that a connected quasi-smooth K-analytic curve C is an open pseudo-annulus *if $C\hat{\otimes}_K\widehat{K^{\mathrm{alg}}}$ is a disjoint union of open pseudo-annuli and if Gal(K^{alg}/K) preserves the orientation of their skeletons. In this case, one can check that C has no boundary, contains no points of positive genus and that its analytic skeleton Γ_C is an open segment. We call Γ_C the skeleton of C.*

Let X be a K-analytic space that is either a virtual open disk or an open pseudo-annulus. Let $(\mathscr{F},\nabla)$ be a module with connection on X. Let r be the rank of $\mathscr{F}$.

Let $x \in X$. Let L be a complete algebraically closed valued field containing $\mathscr{H}(x)$. Then X_L is a disjoint union of open disks (when X is a virtual open disk) or a disjoint union of open pseudo-annuli (when X is an open pseudo-annulus) and there exists an L-rational point x' of X_L over x. Let $D_{x'}$ be the connected component of $X_L - \Gamma_{X_L}$ containing x'. It is an open disk and we identify it to some $D(0,R)^-$ by some isomorphism sending x' to 0.

The pull-back $(\mathscr{F}_L,\nabla_L)$ of $(\mathscr{F},\nabla)$ to X_L is still a module with connection of rank r. For $i \in \{1,\dots,r\}$, denote by $\mathcal{R}'_i$ the supremum of the radii of the closed disks centered at 0 on which the differential equation induced by $(\mathscr{F}_L,\nabla_L)$ admits at least $r-i+1$ solutions that are linearly independent over L. It is strictly positive.

Definition 1.1.4. *For $i \in \{1,\dots,r\}$, the i^{th} radius of convergence of $(\mathscr{F},\nabla)$ at x is*

$$\mathcal{R}_i(x,(\mathscr{F},\nabla)) := \frac{\mathcal{R}'_i}{R} \in \,]0,1]. \tag{1.1}$$

It is independent of the choices made.

The total height of the Newton polygon of $(\mathscr{F},\nabla)$ at x is

$$H_r(x,(\mathscr{F},\nabla)) = \prod_{i=1}^{r} \mathcal{R}_i(x,(\mathscr{F},\nabla)). \tag{1.2}$$

Remark 1.1.5. *If X is a virtual open disk, it follows from the definition that we have $\mathcal{R}_r(x,(\mathscr{F},\nabla)) < 1$ for all $x \in X$ if, and only if, we have $\mathcal{R}_r(x,(\mathscr{F},\nabla)) < 1$ for some $x \in X$.*

Similarly, if X is an open pseudo-annulus, we have $\mathcal{R}_r(x,(\mathscr{F},\nabla)) < 1$ for all $x \in X$ if, and only if, we have $\mathcal{R}_r(x,(\mathscr{F},\nabla)) < 1$ for all $x \in \Gamma_X$.

Remark 1.1.6. *It is possible to define the radii of convergence of a module with connection on an arbitrary quasi-smooth K-analytic curve X (see REF). In this case, one has to specify some normalization, which amounts to choosing a so-called pseudo-triangulation of X (see REF).*

In this paper, we do not consider radii of convergence on arbitrary curves but only on virtual open disks and open pseudo-annuli. For such spaces, the definition we stated above coincides with the general definition when the chosen pseudo-triangulation is the empty one.

Let $x \in X \setminus \Gamma_X$. Let y be a point of $X_{\widehat{K^{\mathrm{alg}}}}$ over x. Let E_y be the connected component of $X_{\widehat{K^{\mathrm{alg}}}} \setminus \Gamma_{X_{\widehat{K^{\mathrm{alg}}}}}$ containing y. It is an open disc and we identify it to some $D(0,\rho)^-$. Denoting by $r(y)$

the radius of the point y in $D(0,\rho)^-$ (see REF), we set

$$\rho(x) := \frac{r(y)}{\rho} \in [0,1). \tag{1.3}$$

It is independent of the choices made.

For $x \in \Gamma_X$, we set

$$\rho(x) := 1. \tag{1.4}$$

Definition 1.1.7. *Let $x \in X$. For $i \in \{1,\dots,r\}$, the i^{th} radius of convergence of $(\mathscr{F},\nabla)$ at x is said to be* spectral *(resp.* solvable*) if we have $\mathcal{R}_i(x,(\mathscr{F},\nabla)) \leqslant \rho(x)$ (resp. $\mathcal{R}_i(x,(\mathscr{F},\nabla)) = \rho(x)$).*

The radii of convergence satisfy nice properties. The main result of [**?**] may be stated as follows.

Theorem 1.1.8. *Let $i \in \{1,\dots,r\}$. The map*

$$\mathcal{R}_i(-,(\mathscr{F},\nabla)) \colon X \to \,]0,1] \tag{1.5}$$

is continuous on X and piecewise log-affine with rational slopes on each segment inside X.

Moreover, there exists a locally finite subgraph of X outside of which the map $\mathcal{R}_i(-,(\mathscr{F},\nabla))$ is locally constant. □

In the rest of the text, we will often write $\mathscr{F}$ instead of $(\mathscr{F},\nabla)$. This should lead to no confusion.

1.2. Local irregularity

In this section, we introduce the crucial notion of local irregularity of a module with connection $(\mathscr{F},\nabla)$ on a quasi-smooth K-analytic curve X. The definition involves the slope of the total height of the convergence Newton polygon along certain germs of segments in X.

The importance of this notion relies on the fact that, under appropriate conditions, it controls the finite dimensionality of the de Rham cohomology groups.

Recall that we have define a notion of germ of segment in X in REF. Note that, according to our conventions, a germ of segment out of a point is always oriented out of that point, while a germ of segment at the open boundary of X is always oriented towards the interior of X.

Definition 1.2.1. *A germ of segment in X is said to be good if it may be represented by the skeleton of an open pseudo-annulus contained in X.*

Remark 1.2.2. *i) Every good germ has finite degree (in the sense of [PP13, Definition 1.1.20]).*

ii) Every germ of segment out of a point is good.

iii) A germ of segment it good if, and only if, it admits a representative on which the map $x \mapsto \deg(x)$ is constant (see [PP13, Lemma 1.1.28]).

iv) A relatively compact germ of segment is good (see [PP13, Lemma 1.1.35]).

Christol and Mebkhout gave a definition of irregularity for solvable differential modules over a germ of open annulus (see [CM00, Définition 8.3-8] and [CM01, Section 2.1]). We here extend this definition to the case of a germ of pseudo-annulus and to possibly non-solvable differential equations whose radii are all log-affine.

Definition 1.2.3 (Log-affine total height over a germ of segment)**.** *Let b be a good germ of segment in X. Let r be the rank of $\mathscr{F}$ around b. Let $i \in \{1,\dots,r\}$. We say that $\mathscr{F}$ has log-affine i^{th} radius (resp. total height) along b if there exists an open pseudo-annulus C in X that whose skeleton* *(suitably oriented) represents b such that the i^{th} radius function $\mathcal{R}_i(-,\mathscr{F}_{|C})$ (resp. the total height function $H_r(-,\mathscr{F}_{|C})$) is log-affine on Γ_C.*

Remark 1.2.4. *If K is trivially valued, then all the radii of $\mathscr{F}$ are log-affine along any good germ of segment (see REF). In particular, $\mathscr{F}$ has log-affine total height along any good germ of segment.*

Definition 1.2.5 (Irregularity over a germ of segment)**.** *Let b be a good germ of segment in X on which $\mathscr{F}$ has log-affine total height. Let C be an open pseudo-annulus whose skeleton Γ_C (suitably oriented) represents b and such that the total height function $H_r(-,\mathscr{F}_{|C})$ is log-affine on Γ_C, where $r = \mathrm{rank}(\mathscr{F}_{|C})$. We define the irregularity of $\mathscr{F}$ along b as*

$$\mathrm{Irr}_b(\mathscr{F}) \;:=\; -\deg(b)\cdot \partial_b H_r(-,\mathscr{F}_{|C}) \;\in\; \mathbb{Z}\,. \tag{1.6}$$

The fact that the irregularity is an integer comes from [PP13, Proposition 2.3.6]. It can be *negative.*

Lemma 1.2.6. *Let b be a good germ of segment in X on which $\mathscr{F}$ has log-affine total height. Let L be a complete valued extension of K. Let $c_1,\ldots,c_t$ be the preimages of b in X_L. Then, the c_i's are good, $\mathscr{F}_L$ has log-affine total height on them and we have*

$$\mathrm{Irr}_b(\mathscr{F}) \;=\; \sum_{i=1}^{t} \mathrm{Irr}_{c_i}(\mathscr{F}_L)\,. \tag{1.7}$$

□

Remark 1.2.7. *We will see that, if $\mathscr{F}$ has a meromorphic singularity at a rational point x and b_x is the germ of segment out of x (oriented out of x), then $\mathscr{F}$ has* log-affine *radii on b_x (cf. Lemma 3.2.3). We will also prove that, in this case, the irregularity $\mathrm{Irr}_{b_x}(\mathscr{F})$ just defined coincides with the opposite of the formal irregularity of $\mathscr{F}$ at x viewed as a differential equation over the field of power series $K((T-x))$ as defined in [Ram78], [Del70, p.110], [Mal74] and [DMR07] (cf. Proposition 3.4.1). See Section 3 for more details.*

In the context of [CM01], our irregularity coincides with that defined therein.

Proposition 1.2.8. *Let $0\to\mathscr{F}_1\to\mathscr{F}_2\to\mathscr{F}_3\to 0$ be an exact sequence of differential equations on a quasi-smooth K-analytic curve X. Let b be a good germ of segment in X. Then $\mathscr{F}_2$ has log-affine total height along b if and only if so have $\mathscr{F}_1$ and $\mathscr{F}_3$. In this case, one has*

$$\mathrm{Irr}_b(\mathscr{F}_2) \;=\; \mathrm{Irr}_b(\mathscr{F}_1) + \mathrm{Irr}_b(\mathscr{F}_3)\,. \tag{1.8}$$

Proof. Definition 1.2.5 of irregularity at a germ of segment involves only spectral radii because in (1.6) one localizes before computing the slope. In particular, it follows from the spectral definition of the radii (cf. [Ked10, Definition 9.8.1] or [Pul15, Section 4.2]), that the localized radii of $\mathscr{F}_2$ are the union with multiplicity of those of $\mathscr{F}_1$ and $\mathscr{F}_3$. In particular the localized total height of $\mathscr{F}_2$ is the logarithmic sum of those of $\mathscr{F}_1$ and $\mathscr{F}_3$. Moreover, by point i) of Theorem **??** REF below the localized total heights (1.6) are log-concave along b, and hence the total height of $\mathscr{F}_2$ is log-affine along b if, and only if, so are the total heights of $\mathscr{F}_1$ and $\mathscr{F}_3$. □

1.3. Polygons and their derivatives on curves.

Let $n\leqslant m\in\mathbb{Z}$. Let $v_n, v_m\in\mathbb{R}$ and, for each $i\in\{n+1,\ldots,m-1\}$, let $v_i\in\mathbb{R}\cup\{\pm\infty\}$. If all the v_i's are different from $-\infty$, we define the Newton polygon of the set

$$V \;=\; \{(i,v_i)\mid n\leqslant i\leqslant m\} \tag{1.9}$$

to be the biggest convex function

$$NP_V \ :\ [n,m] \ \longrightarrow \ \mathbb{R} \tag{1.10}$$

such that, for each $i \in \{n,\dots,m\}$, we have

$$v_i = +\infty \ \text{ or } \ v_i \geqslant NP_V(i)\ . \tag{1.11}$$

If all the v_i's are different from $+\infty$, we define analogously the inverted Newton polygon as the smallest concave function $INP_V \colon [n,m] \to \mathbb{R}$ such that for each $i \in \{n,\dots,m\}$, we have

$$v_i = -\infty \ \text{ or } \ v_i \leqslant INP_V(i)\ . \tag{1.12}$$

The functions NP_V and INP_V are continuous on $[n,m]$ and affine on each $[i,i+1]$.

Definition 1.3.1. *Let N denote either the polygon NP_V or INP_V. For each $i \in \{1,\dots,m-n\}$, we call i^{th} slope of the polygon the slope of the function N on $[n+i-1,n+i]$. We usually denote it by*

$$s_i \ :=\ \frac{d}{du}N(u)\ , \quad u \in [n+i-1,n+i]\ . \tag{1.13}$$

The total height of the polygon is defined as $\sum_{i=1}^{m-n} s_i$.

The polygon is said to have a break at $k \in \{1,\dots,m-n-1\}$ if $s_k \neq s_{k+1}$.

Note that, conversely, given a non-decreasing (resp. non-increasing) sequence of real numbers $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_{m-n}$ (resp. $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_{m-n}$) and a real number v_n, we can define a Newton polygon (resp. inverted Newton polygon) as the unique continuous function on $[n,m]$ that takes the value v_n at n and that is affine of slope s_i over $[n+i-1,n+i]$, for each $i \in \{1,\dots,m-n\}$.

Notation 1.3.2. *In the sequel, if no specific data are mentioned, we assume that $n=0$ and $v_0=0$. In this case the polygon and the inverted polygon are determined by the sequence of their slopes.*

Let X be a quasi-smooth K-analytic curve and let $(\mathscr{F},\nabla)$ be a module with connection on X. Let b be a good germ of segment on X. In this setting, we would like to define a convergence Newton polygon of $(\mathscr{F},\nabla)$ and its derivative along b.

Let C be an open pseudo-annulus whose skeleton (suitably oriented) represents b. Let r be the rank of $\mathscr{F}$ on C. Let $x \in C$.

Definition 1.3.3 (Convergence Newton polygon)**.** *The* convergence Newton polygon of $\mathscr{F}$ at x *is the unique continuous piecewise-affine map $NP(x,\mathscr{F}) : [0,r] \to \mathbb{R}$ satisfying*

i) $NP(x,\mathscr{F})(0) = 0$;

ii) for each $i \in \{1,\dots,r\}$, the i^{th} slope of $NP(x,\mathscr{F})$ is

$$s_i(x) \ :=\ \ln(\mathcal{R}_i(x,\mathscr{F})) \leqslant 0\ . \tag{1.14}$$

In order to simplify the notation, we sometimes set

$$v_j(x) \ :=\ NP(x,\mathscr{F})(j) \in \mathbb{R}\ . \tag{1.15}$$

Assume that the radii $\mathcal{R}_i(-,\mathscr{F})$ are log-affine along b. There are essentially two ways to construct a polygon as a derivative of the convergence Newton polygon along b. The first takes the derivative of the functions (1.15) along b, while the second takes the derivatives of the slopes (1.14):

i) We consider both the *polygon* and the *inverted polygon* associated to the set

$$\{(i,\partial_b v_i)\ ,\ 0 \leqslant i \leqslant r\}\ . \tag{1.16}$$

ii) We consider the family $(\partial_b(s_i))_{i=1,\dots,r}$ of derivatives along b of the slopes (1.14) of the convergence Newton polygon. In general the sequence $(\partial_b(s_1),\dots,\partial_b(s_r))$ is not monotonous. Therefore, we consider a permutation of the indexes σ (resp. σ') that turns it into non-decreasing (resp. non-increasing) order

$$\partial_b(s_{\sigma(1)}) \;\leqslant\; \partial_b(s_{\sigma(2)}) \;\leqslant\; \dots \;\leqslant\; \partial_b(s_{\sigma(r)}) \tag{1.17}$$

$$(\text{resp. } \partial_b(s_{\sigma'(1)}) \;\geqslant\; \partial_b(s_{\sigma'(2)}) \;\geqslant\; \dots \;\geqslant\; \partial_b(s_{\sigma'(r)}))\;, \tag{1.18}$$

and we consider the polygon on $[0,r]$ that takes the value 0 at 0 and that has (1.17) (resp. (1.18)) as slopes.

Remark 1.3.4. *The four polygons defined above all have $-\mathrm{Irr}_b(\mathscr{F})$ as total height. Indeed, their total height is the derivative of $v_r=\sum_{i=1}^r s_i$ which is the total height of the convergence Newton polygon.*

Remark 1.3.5. *We here illustrate with an example the necessity of considering both the polygon and the inverted polygon.*

Let $x\in X$. The finiteness of the radii (see Theorem 1.1.8) shows that for almost all directions b out of x the radii are all constant, therefore both the polygon and the inverted polygon associated with the set $\{(i,\partial_b v_i)\;,\;0\leqslant i\leqslant r\}$ are the zero functions on $[0,r]$. There are a finite number of directions out of x on which the polygons may be non zero.

Assume now that x lies inside an open segment $]z,y[\,\subseteq X$ where the radii are log- affine. Let b_z and b_y be the germs of segments out of x directed towards z and y respectively. Then

$$\partial_{b_z} v_i \;=\; -\partial_{b_y} v_i\;. \tag{1.19}$$

This shows that in order to have the same polygon on b_z and b_y we have to consider the polygon associated with the set $\{(i,\partial_{b_z}v_i)\;,\;0\leqslant i\leqslant r\}$ and the inverted polygon associated with the set $\{(i,\partial_{b_y}v_i)\;,\;0\leqslant i\leqslant r\}$. The change of orientation turns the polygon into the inverted polygon, and it seems unnatural to fix a choice.

1.4. de Rham cohomology

Let X be a quasi-smooth K-analytic curve and let $(\mathscr{F},\nabla)$ be a module with connection on X. Consider the complex of sheaves

$$\mathcal{E}(\mathscr{F})^\bullet\colon (\cdots\to 0\to\mathscr{F}\xrightarrow{\nabla}\Omega^1_X\otimes\mathscr{F}\to 0\to\cdots)\;, \tag{1.20}$$

where $\mathscr{F}$ is placed in degree 0 and $\Omega^1_X\otimes\mathscr{F}$ in degree 1. The cohomology of $\mathscr{F}$ (resp. the hypercohomology of $\mathcal{E}(\mathscr{F})^\bullet$) will be denoted by $H^i(X,\mathscr{F})$ (resp. $\mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^\bullet)$).

Remark 1.4.1. *In our situation, X is a K-analytic curve, hence has topological dimension 1. It follows that $H^i(X,\mathscr{F})=0$ for $i\geqslant 2$ and that $\mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^\bullet)=0$ for $i\geqslant 3$ (by a spectral sequence argument).*

Definition 1.4.2. *The de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ of $\mathscr{F}$ are the hypercohomology groups $\mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^\bullet)$ of the complex $\mathcal{E}(\mathscr{F})^\bullet$:*

$$\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \;:=\; \mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^\bullet)\;. \tag{1.21}$$

We say that $\mathscr{F}$ has finite index *if $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ is finite-dimensional for all degrees $i\in\mathbb{Z}$. In this* *case we denote by* $\mathrm{h}^i_{\mathrm{dR}}(X,\mathscr{F})$ *the dimension of the* K*-vector space* $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ *and set*

$$\chi_{\mathrm{dR}}(X,\mathscr{F}) := \sum_i (-1)^i \cdot \mathrm{h}^i_{\mathrm{dR}}(X,\mathscr{F}) \tag{1.22}$$

$$= \mathrm{h}^0_{\mathrm{dR}}(X,\mathscr{F}) - \mathrm{h}^1_{\mathrm{dR}}(X,\mathscr{F}) + \mathrm{h}^2_{\mathrm{dR}}(X,\mathscr{F}) . \tag{1.23}$$

We call $\chi_{\mathrm{dR}}(X,\mathscr{F})$ *the index of* $\mathscr{F}$*.*

Lemma 1.4.3 (Mayer-Vietoris)**.** *Let* U *and* V *be two open subsets of* X *such that* $X = U \cup V$*. Let* $\mathcal{E}^\bullet$ *be a complex of sheaves of groups over* X*. We have the Mayer-Vietoris long exact sequence*

$$\cdots \to \mathbb{H}^{i-1}(U\cap V,\mathcal{E}^\bullet) \to \mathbb{H}^i(X,\mathcal{E}^\bullet) \to \mathbb{H}^i(U,\mathcal{E}^\bullet)\oplus\mathbb{H}^i(V,\mathcal{E}^\bullet) \to \mathbb{H}^i(U\cap V,\mathcal{E}^\bullet) \to \mathbb{H}^{i+1}(X,\mathcal{E}^\bullet) \to \cdots \tag{1.24}$$

In particular, if, for all $i \in \mathbb{Z}$*, the spaces* $\mathbb{H}^i(U,\mathcal{E}^\bullet)$*,* $\mathbb{H}^i(V,\mathcal{E}^\bullet)$ *and* $\mathbb{H}^i(U\cap V,\mathcal{E}^\bullet)$ *are finite-dimensional, then, for all* $i \in \mathbb{Z}$*, the space* $\mathbb{H}^i(X,\mathcal{E}^\bullet)$ *is finite-dimensional too.* □

1.4.1. Properties In some cases, the de Rham cohomology may be computed in a simple way. Let us first recall a definition.

Definition 1.4.4 (Cohomologically Stein)**.** *The curve* X *is said to be* cohomologically Stein *if, for every coherent sheaf* $\mathscr{F}$ *on* X *and every* $q \geqslant 1$*, we have*

$$H^q(X,\mathscr{F}) = 0. \tag{1.25}$$

Classical examples of cohomologically Stein curves include disks, pseudo-disks, annuli, pseudo-annuli, etc. More generally, by [**?**, Corollary 4.6], every quasi-smooth curve with no proper connected components is cohomologically Stein. We also recall that, on quasi-Stein curves, coherent sheaves are generated by their global sections (see [**?**, Corollary 4.8]) and that the global sections functor induces an equivalence between the category of locally free sheaves of bounded rank and the category of projective $\mathscr{O}(X)$-modules of finite type (see [**?**, Corollary 4.11]).

Lemma 1.4.5. *If* X *is cohomologically Stein, then we have*

$$\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F}) = \mathrm{Ker}(\nabla\colon \mathscr{F}(X) \to \Omega^1(X)\otimes_{\mathscr{O}(X)}\mathscr{F}(X)) \tag{1.26}$$

and

$$\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) = \mathrm{Coker}(\nabla\colon \mathscr{F}(X) \to \Omega^1(X)\otimes_{\mathscr{O}(X)}\mathscr{F}(X)) , \tag{1.27}$$

and $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) = 0$ *for all* $i \neq 0,1$*.* □

Lemma 1.4.6. *If* X *has finitely many connected components, then* $\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F})$ *is finite-dimensional.*

Proof. If X is cohomologically Stein, then the result follows from Lemma 1.4.5 and [PP13, Lemma 1.2.10 i)].

If X is not cohomologically Stein, then we may cover it by two open subsets U and V that are cohomologically Stein. The result then follows from the previous case together with the Mayer-Vietoris exact sequence (see Lemma 1.4.3). □

Let us recall a descent statement for de Rham cohomology that will be used several times in the paper.

Theorem 1.4.7 ([**?**, Corollary 4.14])**.** *Let* L *be a complete valued extension of* K*. Assume that there exists* $M \in \{K,L\}$ *such that* M *is not trivially valued and* $\mathrm{H}^1_{\mathrm{dR}}(X_M,\mathscr{F}_M)$ *is finite-dimensional.*

Then, $\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F})$ *and* $\mathrm{H}^1_{\mathrm{dR}}(X_L,\mathscr{F}_L)$ *are both finite-dimensional and we have natural isomorphisms*

$$\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F})\otimes_K L \xrightarrow{\sim} \mathrm{H}^0_{\mathrm{dR}}(X_L,\mathscr{F}_L) \quad \text{and} \quad \mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F})\otimes_K L \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}(X_L,\mathscr{F}_L). \tag{1.28}$$

□

For later use, we record here some surjectivity result in de Rham cohomology.

Lemma 1.4.8 ([?, Lemma 4.15])**.** *Assume that X has no proper connected component. Let W be an analytic domain of X such that the restriction map $\mathscr{O}(X)\to\mathscr{O}(W)$ has dense image. Assume that there exists a complete non-trivially valued extension L of K such that $\mathrm{H}^1_{\mathrm{dR}}(W_L,(\mathscr{F}_L)_{|W_L})$ is finite-dimensional. Then, the map*

$$\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(W,\mathscr{F}_{|W}) \tag{1.29}$$

is surjective. □

To finish this section, we provide some conditions to ensure that we have trivial cohomology.

Proposition 1.4.9. *Assume that we are in one of the following two situations.*

Situation 1:

i) X is an open pseudo-annulus;

ii) all the radii of $\mathscr{F}$ are log-affine along Γ_X and strictly smaller than 1.

Situation 2:

i) X is a virtual open disk;

ii) all the radii of $\mathscr{F}$ are constant on X and strictly smaller than 1.

Then, for all i, we have

$$\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \;=\; 0\,. \tag{1.30}$$

Proof. Let r be the rank of $\mathscr{F}$. First note that, in situation 1, all the radii of $\mathscr{F}$ are locally constant on $X\setminus\Gamma_X$ by [PP13, Corollary 6.2.28].

We proceed in two steps.

Step 1: Assume that Ω^1_X is free.

If $\mathscr{F}$ admits a global non-zero solution on X, then there exists $x\in X$ such that $\mathcal{R}_r(x,\mathscr{F})=1$. By Remark 1.1.5, this contradicts the assumptions. We deduce that $\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F})=0$.

Since X is cohomologically Stein and Ω^1_X is free, by Lemma 1.4.14, we have $\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) = \mathrm{H}^1_{\mathrm{dR}}(\mathscr{F}(X),\nabla)$. By [Ked10, Lemma 5.3.3 and Remark 5.3.4], we deduce that $\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) = \mathrm{Ext}^1(\mathscr{F}^*(X),\mathscr{O}(X))$. We will now prove that this last group is 0 by proving that every exact sequence of differential modules

$$0\to\mathscr{O}(X)\to E\to\mathscr{F}^*(X)\to 0 \tag{1.31}$$

splits. Since X is cohomologically Stein, the coherent sheaves on X are generated by their global sections, hence such a sequence induces an exact sequence of differential equations

$$0\to\mathscr{O}_X\to\mathscr{E}\to\mathscr{F}^*\to 0\,. \tag{1.32}$$

It is enough to prove that this last sequence splits.

Let us first prove that all the radii of $\mathscr{F}^*$ are everywhere strictly smaller than 1. Let us recall that, by REF, spectral non-solvable radii are invariant under duality. In situation 1, the radii of $\mathscr{F}$ are alll spectral non-solvable on Γ_X by assumption, hence so are the radii of $\mathscr{F}^*$. We then conclude by Remark 1.1.5. In situation 2, by assumption, there exists a point of X on which the radii are all spectral non-solvable and the same reasoning applies.

We have now proven that all the radii of $\mathscr{F}^*$ are everywhere strictly smaller than 1. We also know that all the radii of $\mathscr{O}$ are constant and equal to 1. By [PP13, Proposition 2.10.5], the family of radii of $\mathscr{E}$ is the union (with multiplicities) of those of $\mathscr{F}^*$ and those of $\mathscr{O}_X$. By [PP13, Theorem 5.4.11], we deduce that $\mathscr{E}$ splits into $\mathscr{E}_{<r+1}$ and $\mathscr{E}_{\geqslant r+1}$. By uniqueness, we have $\mathscr{E}_{\geqslant r+1} = \mathscr{O}_X$ and $\mathscr{E}_{<r+1} = \mathscr{F}^*$. The result follows.

Step 2: The general case.

By Theorem 1.4.7, it is enough to prove the result after extending the scalars. Hence, we may assume that K is algebraically closed and maximally complete and that $|K| = \mathbb{R}_{\geqslant 0}$.

If we are in situation 2, then X is a disk and Ω^1_X is free by REF(Lazard), so we can conclude by step 1.

Let us now assume that we are in situation 1. Recall that there exists a deformation retraction $r\colon X \to \Gamma_X$ and that, for each compact interval I in Γ_X, the preimage $r^{-1}(I)$ of I is a closed annulus with analytic skeleton I (see REF). Moreover, the ring of global functions of a closed annulus being principal, the restriction of Ω^1_X to such a space is free. We deduce that, for each open relatively compact interval J, the preimage $r^{-1}(J)$ of J is an open annulus and that the restriction of Ω^1_X to it is free.

Let us now write $\Gamma_X = J_1 \cup J_2$, where J_1 and J_2 are disjoint unions of open relatively compact intervals. Note that $J_1 \cap J_2$ is also a disjoint union of open relatively compact intervals. Set $U_1 := r^{-1}(J_1)$ and $U_2 := r^{-1}(J_2)$. It follows from the previous discussion and from step 1 that, for $W \in \{U, V, U \cap V\}$, we have $\mathrm{H}^0_{\mathrm{dR}}(W, \mathscr{F}) = \mathrm{H}^1_{\mathrm{dR}}(W, \mathscr{F}) = 0$. We conclude by the Mayer-Vietoris long exact sequence. □

Corollary 1.4.10. *Assume that X is a virtual open disk and that the radii of $\mathscr{F}$ are all constant over D. Then $\mathscr{F}$ has finite index on D and we have $\mathrm{H}^1_{\mathrm{dR}}(D, \mathscr{F}) = 0$.*

Proof. By Theorem 1.4.7, we may extend the scalars and assume that K is algebraically closed. The radii of $\mathscr{F}$ are all constant on D. If they are all strictly smaller than 1, then we are in situation 2 of Proposition 1.4.9 and the result holds.

Otherwise, let $j \in \{1, \dots, r\}$ be the smallest index such that $\mathcal{R}_j(-, \mathscr{F}) = 1$ on D. By [PP13, Theorem 5.3.1], there exists a sub-object $\mathscr{F}_{\geqslant j}$ of $\mathscr{F}$ of rank $r - j + 1$ such that, for each $x \in D$, we have

$$\begin{cases} \forall i \in \{1, \dots, r-j+1\},\ \mathcal{R}_i(x, \mathscr{F}_{\geqslant j}) = 1; \\ \forall i \in \{1, \dots, j-1\},\ \mathcal{R}_i(x, \mathscr{F}/\mathscr{F}_{\geqslant j}) = \mathcal{R}_i(x, \mathscr{F}_{|D}) < 1. \end{cases}$$

By Proposition 1.4.9, we have $\mathrm{H}^1_{\mathrm{dR}}(D, \mathscr{F}/\mathscr{F}_{\geqslant j}) = 0$. Since all the radii of the module $\mathscr{F}_{\geqslant j}$ are equal to 1, it is a finite sum of trivial modules. Since the usual derivation is surjective on $\mathscr{O}(D)$, we have $\mathrm{H}^1_{\mathrm{dR}}(D, \mathscr{O}) = 0$, hence $\mathrm{H}^1_{\mathrm{dR}}(D, \mathscr{F}_{\geqslant j}) = 0$. The result now follows by writing the cohomology long exact sequence associated to the short exact sequence $0 \to \mathscr{F}_{\geqslant j} \to \mathscr{F} \to \mathscr{F}/\mathscr{F}_{\geqslant j} \to 0$. □

1.4.2. Differential modules

We can also define Rham cohomology in the setting of differential modules.

Definition 1.4.11 (General definition of index)**.** *Let V be a K-vector space and let $u : V \to V$ be a K-linear map. We say that u has finite index if* $\mathrm{Ker}(u)$ *and* $\mathrm{Coker}(u)$ *are finite-dimensional K-vector spaces. In this case, we define the* index *of u as*

$$\chi(V, u) := \dim_K \mathrm{Ker}(u) - \dim_K \mathrm{Coker}(u)\ . \tag{1.33}$$

Lemma 1.4.12 (Additivity of index)**.** *Let*

$$\begin{tikzcd} 0 \arrow[r] & V_1 \arrow[r] \arrow[d, "u_1"] & V_2 \arrow[r] \arrow[d, "u_2"] & V_3 \arrow[r] \arrow[d, "u_3"] & 0 \\ 0 \arrow[r] & V_1 \arrow[r] & V_2 \arrow[r] & V_3 \arrow[r] & 0 \end{tikzcd} \tag{1.34}$$

be a commutative diagram of K-vector spaces in which the horizontal sequences are both exact. If two among u_1, u_2, u_3 have finite index, then so has the third. In this case, we have

$$\chi(V_2, u_2) \;=\; \chi(V_1, u_1) + \chi(V_3, u_3)\;. \tag{1.35}$$

Proof. This follows from the exact sequence $0 \to \mathrm{Ker}(u_1) \to \mathrm{Ker}(u_2) \to \mathrm{Ker}(u_3) \to \mathrm{Coker}(u_1) \to \mathrm{Coker}(u_2) \to \mathrm{Coker}(u_3) \to 0$ given by the snake lemma. □

Definition 1.4.13 (de Rham cohomology of a differential module)**.** *Let A be a K-algebra together with a derivation $d\colon A \to A$ such that $K = \mathrm{Ker}(d)$. Let (M, ∇) be a differential module over (A, d), i.e. an A-module M equipped with a connection $\nabla\colon \mathrm{M} \to \mathrm{M}$ satisfying the Leibniz rule:*

$$\forall a \in A, m \in M,\;\; \nabla(am) = d(a)m + a\nabla(m). \tag{1.36}$$

If ∇ has finite index, we set

$$\mathrm{H}^0_{\mathrm{dR}}(\mathrm{M}, \nabla) \;:=\; \mathrm{Ker}(\nabla), \tag{1.37}$$

$$\mathrm{H}^1_{\mathrm{dR}}(\mathrm{M}, \nabla) \;:=\; \mathrm{Coker}(\nabla), \tag{1.38}$$

$$\chi_{\mathrm{dR}}(\mathrm{M}, \nabla) \;:=\; \chi(\mathrm{M}, \nabla)\;. \tag{1.39}$$

Lemma 1.4.14. *If X is cohomologically Stein and if the sheaf Ω^1_X is free of rank 1, then Definitions 1.4.2 and 1.4.13 can be made to agree. Namely, choose a global differential form on X that is a basis of $\Omega^1_X(X)$ and consider the associated derivation d. Then, the differential equation $(\mathscr{F}, \nabla)$ induces a differential module $(\mathscr{F}(X), \nabla)$ over $(\mathscr{O}(X), d)$ and, for $i = 0, 1$, we have*

$$\mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F}) \;=\; \mathrm{H}^i_{\mathrm{dR}}(\mathscr{F}(X), \nabla)\;, \tag{1.40}$$

$$\chi_{\mathrm{dR}}(X, \mathscr{F}) \;=\; \chi_{\mathrm{dR}}(\mathscr{F}(X), \nabla)\;. \tag{1.41}$$

□

Remark 1.4.15. *The index $\chi(\mathscr{F}(X), \nabla)$* depends on the chosen derivation of $\mathscr{O}(X)$. *Namely if fd, is another derivation, with $f \in \mathscr{O}(X)$, then $(\mathscr{F}(X), f\nabla)$ is a differential module over $(\mathscr{O}(X), fd)$. Hence with the notations of Definition 1.4.11 we have*

$$\chi(\mathscr{F}(X), f\nabla) \;=\; \chi(\mathscr{F}(X), f) + \chi(\mathscr{F}(X), \nabla)\;. \tag{1.42}$$

The equality $\chi(\mathscr{F}(X), \nabla) = \chi_{\mathrm{dR}}(X, \mathscr{F})$ holds if and only if d generates Ω^1_X.

1.5. Meromorphic de Rham cohomology.

In this section, we introduce definitions and basic results on meromorphic differential equations.

Let P be a quasi-smooth K-analytic curve. Let Z be a locally finite subset of rigid points of P. Set

$$Y \;:=\; P - Z\;. \tag{1.43}$$

We denote by $j\colon Y \hookrightarrow P$ the associated open immersion. We denote by $\mathscr{O}_P[*Z]$ the sheaf of meromorphic functions on P that are holomorphic on Y (hence have poles at worst on Z). Recall that it is the sheaf on P associated to the presheaf whose ring of sections on an analytic domain U of P is the localization of $\mathscr{O}_P(U)$ by the subset of its elements that do not vanish outside Z.

We now define meromorphic connections following [HTT08, Chapter 5] (which itself borrows from [Del70]).

Definition 1.5.1. *Let $\mathcal{F}$ be a locally free $\mathscr{O}_P[*Z]$-module of finite rank on P. A* meromorphic connection on $\mathcal{F}$ with poles on Z *is a K-linear map*

$$\nabla\colon \mathcal{F} \to \Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F} \tag{1.44}$$

that satisfies the Leibniz rule: for every open subset U of P and every $f \in \mathscr{O}_P[*Z](U)$ and $s \in \mathcal{F}(U)$, we have[1]

$$\nabla(fs) = df \otimes s + f\nabla s. \tag{1.45}$$

We also say that the pair $(\mathcal{F}, \nabla)$ is a differential equation on $P(*Z)$ *or a* (meromorphic) differential equation on P with poles on Z. *As usual, morphisms of differential equations $\varphi\colon (\mathcal{F}, \nabla) \to (\mathcal{F}', \nabla')$ are morphisms of $\mathscr{O}_P[*Z]$-modules that are compatible with the connections.*

Definition 1.5.2. *Let $(\mathcal{F}, \nabla)$ be a differential equation on $P(*Z)$. The de Rham cohomology groups*

$$\mathrm{H}^i_{\mathrm{dR}}(P(*Z), (\mathcal{F}, \nabla)) \tag{1.46}$$

of $(\mathcal{F}, \nabla)$ are the hypercohomology groups of the complex

$$\cdots \to 0 \to \mathcal{F} \xrightarrow{\nabla} \Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F} \to 0 \to \cdots\,, \tag{1.47}$$

where $\mathcal{F}$ is placed in degree 0 and $\Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F}$ in degree 1.

As usual, we will often suppress ∇ from the notation when it is clear from the context.

The notation $\mathscr{F}$ will be used to indicate the restriction of $\mathcal{F}$ to Y:

$$\mathscr{F} \;:=\; \mathcal{F}_{|Y}\,. \tag{1.48}$$

This operation gives rise to a functor

$$\begin{tikzcd} \{\text{Differential equations on } P(*Z)\} \arrow[d, "\mathcal{F} \mapsto \mathcal{F}_{|Y} = \mathscr{F}"] \\ \{\text{Analytic differential equations on } Y\} \end{tikzcd} \tag{1.49}$$

and a canonical morphism between the cohomology groups

$$\mathrm{H}^i_{\mathrm{dR}}(P(*Z), \mathcal{F}) \;\to\; \mathrm{H}^i_{\mathrm{dR}}(Y, \mathscr{F})\,. \tag{1.50}$$

When we do not mention poles, we understand that the connection is holomorphic: $Z = \emptyset$ and $\mathcal{F} = \mathscr{F}$ is a genuine analytic differential equation over $Y = P$.

Remark 1.5.3. *If U is an open subset of P such that $U \cap Z = \emptyset$, the restriction of the sheaf $\mathscr{O}_P[*Z]$ to U coincides by definition with $\mathscr{O}_U$. Hence, over U, we have the usual* analytic *cohomology:*

$$\mathrm{H}^i_{\mathrm{dR}}(U(*Z), \mathcal{F}_{|U}) \;=\; \mathrm{H}^i_{\mathrm{dR}}(U, \mathscr{F}_{|U})\,. \tag{1.51}$$

Lemma 1.5.4. *If the valuation of K is trivial, for each open subset U of P one has*

$$\mathscr{O}_P[*Z](U) \;=\; \mathscr{O}_{P-Z}(U - Z)\,. \tag{1.52}$$

In particular, the restriction functor (1.49) is an equivalence of categories.

[1] Remark that it is enough to require that (1.45) holds for $f \in \mathscr{O}_P(U)$.

Proof. The equality can be tested on a basis of open subsets of P. By Remark 1.5.3, it is enough to test the equality on a basis of open neighborhoods of a point $z \in Z$, which is given by the set of virtual open disks containing z. Therefore, we can assume that P is a virtual open disk and that Z is reduced to an individual rigid point z. Up to replacing K by a finite extension, we can assume that P is an open disk and that z is a K-rational point. In this case the equality $\mathscr{O}_{P-\{z\}} = \mathscr{O}_P[*z]$ follows from an explicit computation (cf. Section 2.5 for more details). □

2. The analytic index theorem over open pseudo-annuli

In this section, we consider a differential equation over an open pseudo-annulus with log-affine radii and we provide a necessary and sufficient condition for the finite dimensionality of its de Rham cohomolgy. This condition is expressed by the finiteness of certain *absolute indexes* at the open boundary of the pseudo-annulus. Absolute indexes are normalized versions of the *generalized indexes* introduced by Robba [Rob84, Rob85] and also exploited by Christol and Mebkhout [CM00]. We improve their approach by the introduction of an intrinsic definition of absolute index which is independent of the coordinate, of the choice of the derivation and of the basis of the differential module. More specifically, this section generalizes the analogous results of [CM00] where similar results were obtained with some restrictions, in particular about the ground field K. As an example, the investigation of the case where K is trivially valued permits to establish the link with the classical theory of differential equations over a field of power series and leads to new proofs of some major well known results in that context (cf. Section 2.5).

We use as a reference the book of P. Schneider [Sch02]. There, the base field is always assumed to be non-trivially valued and spherically complete, which explains why this hypothesis appears in several technical statements. Nevertheless, we have been able to remove it from the major statements thanks to a descent result from [**?**] (see Theorem 1.4.7).

A complete and self-contained exposition turned out to be necessary in this section because several technical parts do not admit analogous accurate statements in [CM01] and will be generalized or used as central tools in Section 3, where we will obtain similar results for differential equations with some meromorphic singularities over an open disk. Indeed, the case of a disk with some meromorphic singularities is a central tool in the proof of our main index results in the forecoming paper [**?**].

2.1. Generalized indexes of vector spaces

Let $\mu \in \mathbb{N}$ and let $V_0, \dots, V_\mu$ be vector spaces over K. Denote by V their direct sum. For every $k \in \{0, \dots, \mu\}$, we denote the canonical injections and projections by

$$i_k \;:\; V_k \hookrightarrow V \qquad \text{and} \qquad p_k \;:\; V \to V_k \tag{2.1}$$

and the corresponding projector by

$$\pi_k \;:=\; i_k \circ p_k \;:\; V \to V\;. \tag{2.2}$$

Recall the notion of operator of finite index from Definition 1.4.11.

Definition 2.1.1. *Let f be an endomorphism of V. For $k \in \{0, \dots, \mu\}$, set $f_k := p_k \circ f \circ i_k$:*

$$f_k \;:\; V_k \xrightarrow{i_k} V \xrightarrow{f} V \xrightarrow{p_k} V_k\;. \tag{2.3}$$

We say that f has finite k-generalized index if f_k has finite index. In this case, the index of the map f_k is denoted by

$$\chi_k^{\mathrm{gen}}(V, f) \;:=\; \chi(V_k, f_k)\;. \tag{2.4}$$

and called the k-generalized index of f.

We will also use a similar notion when each of the V_i's carries a family of seminorms v_i that makes it a normoid Fréchet space (see Appendix **??** and especially Definition **??**). By Section **??**, there exists a family of seminorms v on their direct sum V that makes it a Banachoid space, hence a normoid Fréchet space, and we will consider bounded endomorphisms of (V, v). In the following, we will always use this setting implicitly when writing that f is a bounded endomorphism of V.

Definition 2.1.2. *Let f be a bounded endomorphism of V. It is said to be Fredholm if it is topologically strict and has finite index.*

For $k \in \{0, \dots, \mu\}$, we say that f is k-Fredholm if f_k is Fredholm.

Remark 2.1.3. *Assume that K is not trivially valued. Let V be a Fréchet space and let f be a continuous endomorphism of V. Let v be a family of seminorms on V that induces its topology. Then, by Lemma **??**, f induces a bounded endomorphism of (V, v). Moreover, by Proposition **??**, if f has finite cokernel, then it is topologically strict.*

In other words, in the case where K is not trivially valued, Fredholm operators are nothing but operators of finite indexes.

Lemma 2.1.4 (Descent)**.** *Let f be an endomorphism of V (resp. a bounded endomorphism of V). Let L be an extension of K (resp. a complete valued extension of K). Denote by f_L the endomorphism $f \otimes \mathrm{Id}_L$ of $V_L := V \otimes_K L$ (resp. $V_L := V \hat{\otimes}_K L$).*

Then, f has finite index (resp. is Fredholm) if, and only if, f_L has finite index (resp. is Fredholm) and, in this case, we have

$$\chi(V, f) \;=\; \chi(V_L, f_L)\;. \tag{2.5}$$

For $k \in \{1, \dots, \mu\}$, set $V_{k,L} := V_k \otimes_K L$ (resp. $V_{k,L} := V_k \hat{\otimes}_K L$). The space V_L is naturally isomorphic to the direct sum of the $V_{k,L}$'s (resp. in the category of normoid Fréchet spaces). For $k \in \{1, \dots, \mu\}$, f has finite generalized k-index (resp. is k-Fredholm) with respect to $V = \bigoplus_i V_i$ if, and only if, f_L has finite generalized k-index (resp. is k-Fredholm) with respect to $V_L = \bigoplus_i V_{L,i}$ and, in this case, we have

$$\chi_k^{\mathrm{gen}}(V, f) \;=\; \chi_k^{\mathrm{gen}}(V_L, f_L)\;. \tag{2.6}$$

Proof. In the classical case, the first part is a consequence of the exactness of the tensor product $\cdot \otimes_K L$. In the normoid Fréchet case, it follows from Lemmas **??** and **??** and Proposition **??**.

In the classical case, it is well-known that tensor products commutes with direct sums, hence $V \otimes_K L = \bigoplus_i V_i \otimes_K L$. In the normoid Fréchet case, by Corollary **??**, completed tensor products commute with colimits, hence direct sums. It follows that we also have a direct sum decomposition $V \hat{\otimes}_K L = \bigoplus_i V_i \hat{\otimes}_K L$. The rest of the result is a consequence of the first part. □

Using Remark 2.1.3, we can rephrase the Fredholm property in terms of finite dimensionality of the cokernel after extension of scalars.

Corollary 2.1.5. *Let f be a bounded endomorphism of V. The following conditions are equivalent.*

i) f is Fredholm;

ii) for each complete valued extension L of K, f_L has finite-dimensional kernel and cokernel;

iii) there exists a complete valued extension L of K with non-trivial valuation such that f_L has finite-dimensional kernel and cokernel.

Similar results hold for k-Fredholm. □

Fredholm morphisms satisfy a two out of three principle (see [Sch02, Lemma 22.1]).

Lemma 2.1.6 (Composition)**.** *Let f and g be endomorphisms of V (resp. bounded endomorphisms of V). If two among $f, g, f \circ g$ are of finite index (resp. are Fredholm), then so is the third. In this case, we have*

$$\chi(f \circ g) = \chi(f) + \chi(g). \tag{2.7}$$

Proof. In the classical case, the claim follows from the two exact sequences

$$0 \to \mathrm{Ker}(g) \to \mathrm{Ker}(f \circ g) \xrightarrow{g} \mathrm{Im}(g) \cap \mathrm{Ker}(f) \to 0 \tag{2.8}$$

and

$$0 \to \mathrm{Ker}(f)/(\mathrm{Ker}(f) \cap \mathrm{Im}(g)) \to \mathrm{Coker}(g) \xrightarrow{f} \mathrm{Im}(f)/\mathrm{Im}(f \circ g) \to 0\ . \tag{2.9}$$

In the normoid Fréchet case, by Lemma 2.1.4, we may extend the scalars in order to assume that K is not trivially valued. By Remark 2.1.3, this allows us to forget about topological strictness in the definition of Fredholm operators and we are reduced to the classical case. □

A general notion of *compact* operator is defined in [Sch02, Section 16], with the restriction that the base field is required to be non-trivially valued and maximally complete. At the moment, no definition of compact operator is available outside this case and we propose to get around this difficulty by extending the scalars. Thanks to Lemma 2.1.4, this is harmless for proving index theorems.

Definition 2.1.7 (Potentially compact operator)**.** *We say that a bounded endomorphism c of V is potentially compact if there exists a complete non-trivially valued maximally complete extension L of K such that the endomorphism of $V\hat{\otimes}_K L$ induced by c is compact.*

Remark 2.1.8. *Let c, f, g be bounded endomorphisms of V with c potentially compact. Then $f \circ c \circ g$ is potentially compact. Actually, this holds with "potentially compact" replaced by "compact" when the base field is not trivially valued and maximally complete by [Sch02, Remark 16.7]. Our statement is a straightforward consequence.*

Proposition 2.1.9 (Compact perturbation)**.** *Let f and c be bounded endomorphisms of V. Assume that c is potentially compact.*

Then f is Fredholm if, and only if, $f + c$ is Fredholm and, in this case, we have

$$\chi(V, f) = \chi(V, f + c)\ . \tag{2.10}$$

Proof. Assume that f is Fredholm. By Lemma 2.1.4, we can extend the scalars in order to assume that K is not trivially valued and maximally complete and that c is compact.

By [Sch02, Corollary 22.11], f is invertible modulo compact operators: there exists continuous endomorphisms g, c', c'' of V with c' and c'' compact such that $f \circ g = 1 + c'$ and $g \circ f = 1 + c''$.

Now write $(f + c) \circ g = f \circ g + c \circ g = 1 + c' + c \circ g$. By [Sch02, Remark 16.7], $c' + c \circ g$ is compact and, by [Sch02, Corollary 22.9], $1 + c' + c \circ g$ is Fredholm with index equal to 0. Since $\mathrm{Coker}((f + c) \circ g)$ maps surjectively onto $\mathrm{Coker}(f + c)$, the latter is finite-dimensional.

Similarly, $g \circ (f + c)$ is Fredholm with zero index. Since $\mathrm{Ker}(f + c)$ is contained in $\mathrm{Ker}(g \circ (f + c))$, we deduce that $\mathrm{Ker}(f + c)$ is finite-dimensional and, finally, that $f + c$ is Fredholm (see Remark 2.1.3).

By [Sch02, Corollary 22.9], $1 + c' = f \circ g$ is Fredholm with index 0. By Lemma 2.1.6, we deduce that g is Fredholm and that $\chi(V, f) = -\chi(V, g)$. Using the same argument with $1 + c' + c \circ g = (f + c) \circ g$, we find

$$\chi(V, f + c) = -\chi(V, g) = \chi(V, f). \tag{2.11}$$

To prove the converse, we can add the compact operator $-c$ to $f + c$ and deduce that f is Fredholm from the fact that $f + c$ is Fredholm. $\square$

We now want to establish an analogue of Lemma 2.1.6 for generalized indexes. As the reader may expect, this property fails for general endomorphisms, hence we will focus on a class of endomorphisms satisfying the following compactness property. It will be automatically satisfied by connections (see Proposition 2.2.10).

Definition 2.1.10. *We say that a bounded endomorphism f of V satisfies the* compactness property *if, for every $k, s \in \{0, \dots, \mu\}$ with $s \neq k$, the operator $\pi_s \circ u \circ \pi_k$ is potentially compact.*

Lemma 2.1.11. *Let f and g be bounded endomorphisms of V. Assume that at least one among f and g satisfies the compactness property of Definition 2.1.10.*

Let $k \in \{0, \dots, \mu\}$. If two among $f, g, f \circ g$ are k-Fredholm, then so is the third. In this case, we have

$$\chi_k^{\mathrm{gen}}(V, f \circ g) \;=\; \chi_k^{\mathrm{gen}}(V, f) \;+\; \chi_k^{\mathrm{gen}}(V, g)\;. \tag{2.12}$$

Proof. By Proposition 2.1.9, it is enough to prove that $(f \circ g)_k - f_k \circ g_k$ is a potentially compact endomorphism of V_k.

Consider the bounded endomorphism of V defined by $\alpha := \pi_k \circ f \circ g \circ \pi_k - \pi_k \circ f \circ \pi_k \circ g \circ \pi_k$. Since $(f \circ g)_k - f_k \circ g_k = p_k \circ \alpha \circ i_k$, by Remark 2.1.8, it is enough to prove that α is potentially compact. Observing that $g = \sum_k \pi_k \circ g$, we can write $\alpha = \sum_{s \neq k} \pi_k \circ f \circ \pi_s \circ g \circ \pi_k$ and potential compactness now follows from Remark 2.1.8 again. $\square$

Proposition 2.1.12. *Let f be a bounded endomorphism of V satisfying the compactness property of Definition 2.1.10.*

Then, f is Fredholm if, and only if, for every $k \in \{0, \dots, \mu\}$, f is k-Fredholm. In this case, we have

$$\chi(V, f) \;=\; \sum_{k=0}^{\mu} \chi_k^{\mathrm{gen}}(V, f)\;. \tag{2.13}$$

Proof. We have $f = \sum_{k,s} \pi_k \circ f \circ \pi_s = \sum_k \pi_k \circ f \circ \pi_k + \sum_{k \neq s} \pi_k \circ f \circ \pi_s$. By assumption, for all $k \neq s$, the operator $\pi_k \circ f \circ \pi_s$ is potentially compact. By Proposition 2.1.9, f is Fredholm if, and only if, $\sum_k \pi_k \circ f \circ \pi_k$ is and, in this case, they have the same index. The result now follows from the equality $\sum_k \pi_k \circ f \circ \pi_k = \bigoplus_k f_k$. $\square$

2.2. Generalized indexes for connections over open pseudo-annuli.

2.2.1. Standard pseudo-annuli.

We fix a coordinate T on $\mathbb{A}_K^{1,\mathrm{an}}$. The definitions that follow will depend on it.

Definition 2.2.1. *A K-analytic space C is said to be a standard open pseudo-annulus over K if it is isomorphic to an open subset of $\mathbb{A}_K^{1,\mathrm{an}}$ of the form $\{r_1 < |T| < r_2\}$, where r_1, r_2 are elements of $[0, \infty]$ such that $r_1 < r_2$.*

Let $C = \{r_1 < |T| < r_2\}$, with $r_1, r_2 \in [0, \infty]$ and $r_1 < r_2$, be a standard open pseudo-annulus over K. We consider the open pseudo-disks (cf. [PP13, Definition 1.1.8])

$$D_0 := \{|T| < r_2\}\;, \tag{2.14}$$

and

$$D_1 := \{|T| > r_1\} \cup \{+\infty\} \;, \tag{2.15}$$

that is the complement of $\{|T| \leqslant r_1\}$ in $\mathbb{P}^{1,\mathrm{an}}_K$. Remark that $C = D_0 \cap D_1$. For $k = 0, 1$, we denote by

$$b_k \tag{2.16}$$

the germ of segment at the open boundary of D_k.

We have

$$\mathscr{O}(D_0) \;=\; \{\sum_{n\geqslant 0} a_n T^n \;,\; a_n \in K \;,\; \lim_{n\to+\infty} |a_n|\rho^n = 0 \;,\; \forall \rho < r_2\} \tag{2.17}$$

$$\mathscr{O}(D_1) \;=\; \{\sum_{n\leqslant 0} a_n T^n \;,\; a_n \in K \;,\; \lim_{n\to-\infty} |a_n|\rho^n = 0 \;,\; \forall \rho > r_1\} \tag{2.18}$$

and

$$\mathscr{O}(C) \;=\; \{\sum_{n\in\mathbb{Z}} a_n T^n \;,\; a_n \in K \;,\; \lim_{n\to\pm\infty} |a_n|\rho^n = 0 \;,\; \forall \rho \in]r_1, r_2[\} \;. \tag{2.19}$$

Let us endow $\mathscr{O}(C)$ with an admissible normoid Fréchet structure, for instance the structure associated to an affinoid covering of C (see Definition **??** REF, Lemma **??** and Remark **??**).

Let $m, n \in \mathbb{Z}$ with $m < n$. The spaces $T^m\mathscr{O}(D_1)$, $K \cdot T^{m+1}, \ldots, K \cdot T^{n-1}$ and $T^n\mathscr{O}(D_0)$ of $\mathscr{O}(C)$ are all closed subspaces of $\mathscr{O}(C)$, hence naturally inherit structures of normoid Fréchet spaces. We have a direct sum decomposition in the category of normoid Fréchet spaces

$$\mathscr{O}(C) \;=\; T^m\mathscr{O}(D_1) \oplus \Big(\bigoplus_{s=m+1}^{n-1} K \cdot T^s\Big) \oplus T^n\mathscr{O}(D_0) \;. \tag{2.20}$$

Let V be a free $\mathscr{O}(C)$-module of finite rank r and let j be an $\mathscr{O}(C)$-linear isomorphism

$$j \;:\; V \xrightarrow{\sim} \mathscr{O}(C)^r \;. \tag{2.21}$$

The normoid Fréchet structure on $\mathscr{O}(C)$ induces one on $\mathscr{O}(C)^r$, hence also one on V *via* j.

Remark 2.2.2. *The choice of j does not really affect the normoid structure on V in the sense that another choice would produce an equivalent structure. In particular, the bounded endomorphisms of V remain the same.*

The direct sum decomposition (2.20) induces a direct sum decomposition of V in the category of normoid Fréchet spaces. We denote by

$$V_1 \;:=\; j^{-1}(T^m\mathscr{O}(D_1)^r) \qquad \text{and} \qquad V_0 \;:=\; j^{-1}(T^n\mathscr{O}(D_0)^r) \tag{2.22}$$

the extremal summands of this decomposition.

Definition 2.2.3. *Let f be a bounded endomorphism of V. For $k \in \{0, 1\}$, we say that f is Fredholm at b_k if f_k is Fredholm (see Definition 2.1.2). In this case, we set*

$$\chi^{\mathrm{gen}}_{b_k}(V, f) \;:=\; \chi^{\mathrm{gen}}_k(V, f) \;=\; \chi(V_k, f_k) \tag{2.23}$$

and call this quantity the generalized index of f at b_k.

The notation $\chi^{\mathrm{gen}}_{b_k}$ is justified by the following lemma, which shows that this index is stable by restriction to a sub-annulus of C containing b_k.

Lemma 2.2.4. *Let $k \in \{0,1\}$. Let $C' \subseteq C$ be an open sub-pseudo-annulus such that $b_k \subseteq \Gamma_{C'}$. Endow $\mathscr{O}(C')$ with the normoid Fréchet structure induced by that on $\mathscr{O}(C)$ and $V' := V \otimes_{\mathscr{O}(C)} \mathscr{O}(C')$ with the one given by the tensor product.*

Let f and f' be bounded endomorphisms of V and V' respectively, commuting with the natural restriction $\rho_k : x \in V \mapsto x \otimes 1 \in V'$. Then, f is Fredholm at b_k if, and only if, f' is and, in this case, we have

$$\chi^{\mathrm{gen}}_{b_k}(V, f) \;=\; \chi^{\mathrm{gen}}_{b_k}(V', f') \,. \tag{2.24}$$

Proof. The result follows immediately form the fact that both i_k and p_k commute with ρ_k. □

The definition of $\chi^{\mathrm{gen}}_{b_k}(V, f)$ seems to depend on the choices of n, m, j and the embedding of C into $\mathbb{A}^{1,\mathrm{an}}_K$ (*i.e.* a coordinate on C). We will now show that it does not depend on n, m and j. In the case of connections, we will show later that it is also independent of the choice of a coordinate on C (see Proposition 2.3.12).

Lemma 2.2.5. *Let f be a bounded endomorphism of V. For $b \in \{b_0, b_1\}$, the definition of $\chi^{\mathrm{gen}}_b(V, f)$ is independent of the choices of n and m.*

Moreover, if f satisfies the compactness property of Definition 2.1.10, then f is Fredholm if, and only if, it is Fredholm at b_0 and b_1 and, in this case, one has

$$\chi(V, f) \;=\; \chi^{\mathrm{gen}}_{b_0}(V, f) + \chi^{\mathrm{gen}}_{b_1}(V, f) \,. \tag{2.25}$$

Proof. Let $m', n' \in \mathbb{Z}$ with $m' < n'$. We may assume that $m \leqslant m'$. In this case, $T^m\mathscr{O}(D_1)$ is a subspace of $T^{m'}\mathscr{O}(D_1)$ of finite codimension and the projection of $\mathscr{O}(C)$ onto $T^m\mathscr{O}(D_1)$ factors through $T^{m'}\mathscr{O}(D_1)$. This injection and this projection are both Fredholm and their indexes are opposite. It now follows from Lemma 2.1.6 that f is Fredholm at b_1 with respect to the decomposition associated to n and m if, and only if, it is with respect to the decomposition associated to n' and m' and that, in this case, the generalized indexes at b_1 coincide. The result for b_0 is proved the same way.

The second part of the claim follows from Proposition 2.1.12 and the fact that any operator on a finite-dimensional vector space is Fredholm with zero index. □

Lemma 2.2.6. *Let f be a bounded endomorphism of V that is $\mathscr{O}(C)$-linear. Identify $\det(f)$ with an element of $\mathscr{O}(C)$. Let $b \in \{b_0, b_1\}$. Then, f is Fredholm at b if, and only if, $\det(f)$ has no zeros on some sub-pseudo-annulus of C containing b. Moreover, in this case, the generalized index of f at b is equal to*

$$\chi^{\mathrm{gen}}_b(V, f) \;=\; -\partial_b(\det(f)) \,. \tag{2.26}$$

In particular, $\chi^{\mathrm{gen}}_b(V, f)$ does not depend on the coordinate T of C.

Proof. By Lemma 2.1.4, we may extend the scalars and assume that K is non-trivially valued and maximally complete. In this case, the result follows from [CM00, Proposition 8.2-6] (see also Remark 8.9-10 at the end of Section 8.2 of *ibid.*) or [Tsu98, Section 6]. □

We now prove that the generalized indexes are independent of the choice of j and that they are compatible with exact sequences of $\mathscr{O}(C)$-modules (cf. Lemmas 2.2.7 and 2.2.8). Since we use Lemma 2.1.11, we assume that f satisfies the compactness property of Definition 2.1.10.

Lemma 2.2.7. *Let f be a bounded endomorphism of V that satisfies the compactness property of Definition 2.1.10. Let $b \in \{b_0, b_1\}$. Then, the fact that f is Fredholm at b does not depend on the* *choice of j and, when it is, the value of the generalized index $\chi_b^{\mathrm{gen}}(V,f)$ does not depend on the choice of j either.*

Proof. Let g be an $\mathscr{O}(C)$-linear bounded automorphism of V. By Lemma 2.2.6, it is Fredholm at b. By Lemma 2.1.11, f is Fredholm at b if, and only if, $g\circ f\circ g^{-1}$ is and, in this case, we have

$$\chi_b^{\mathrm{gen}}(V,g\circ f\circ g^{-1})=\chi_b^{\mathrm{gen}}(V,g)+\chi_b^{\mathrm{gen}}(V,f)-\chi_b^{\mathrm{gen}}(V,g)=\chi_b^{\mathrm{gen}}(V,f). \tag{2.27}$$

The result follows. □

Lemma 2.2.8. *Let $0\to U\xrightarrow{\varphi}V\xrightarrow{\psi}W\to 0$ be an exact sequence of finite free $\mathscr{O}(C)$-modules. Let f_U,f_V,f_W be endomorphisms of U,V,W respectively commuting with the maps φ and ψ. Assume that f_U,f_V,f_W satisfy the compactness property of Definition 2.1.10.*

Let $k\in\{0,1\}$. If two operators among f_V,f_U,f_W are Fredholm at b_k, then so is the third. In this case, we have

$$\chi_{b_k}^{\mathrm{gen}}(V,f_V)\;=\;\chi_{b_k}^{\mathrm{gen}}(U,f_U)+\chi_{b_k}^{\mathrm{gen}}(W,f_W)\,. \tag{2.28}$$

Proof. Choose a basis of V that is obtained by putting together a basis of U and a lift of a basis of W and consider the corresponding $\mathscr{O}(C)$-linear isomorphisms j_U,j_V,j_W. Recall that the result does not depend on thoses choices by Lemma 2.2.7.

The exact sequence $0\to U\to V\to W\to 0$ now induces an exact sequence $0\to U_k\to V_k\to W_k\to 0$ that commutes with the truncated operators f_k and g_k (see (2.3)). The claim then follows from Lemma 1.4.12. □

2.2.2. Absolute generalized indexes for connections over standard pseudo-annuli.

We now want to modify the definition of generalized index in order to obtain a notion that is intrinsic in the case of connections.

As before, we consider a standard open pseudo-annulus $C=\{r_1<|T|<r_2\}$ with $0\leqslant r_1<r_2\leqslant+\infty$. We retain the notations of the previous section.

Remark 2.2.9. *Let $b\in\{b_0,b_1\}$. Then d/dT and $T\,d/dT$ are Fredholm at b and we have*

$$\chi_b^{\mathrm{gen}}(\mathscr{O}(C),d/dT)\;=\;\begin{cases}1 & \text{if } b=b_0,\\ -1 & \text{if } b=b_1.\end{cases}\qquad \chi_b^{\mathrm{gen}}(\mathscr{O}(C),T d/dT)\;=\;0\,. \tag{2.29}$$

If $d:\mathscr{O}(C)\to\mathscr{O}(C)$ is a continuous derivation, there exists $h\in\mathscr{O}(C)$ such that $d=h\,d/dT$. By Lemma 2.1.11, d is Fredholm at b if, and only if, so is the multiplication by h (see Lemma 2.2.6).

Let $\nabla:\mathscr{F}\to\Omega^1_C\otimes\mathscr{F}$ be a differential equation over C of rank r. The $\mathscr{O}(C)$-module $\Omega^1(C)$ is free of rank 1 with basis dT. This induces an isomorphism $\ell\colon\Omega^1(C)\xrightarrow{\sim}\mathscr{O}(C)$ given by $\ell(f(T)dT)=f(T)$.

Let $d:\mathscr{O}(C)\to\mathscr{O}(C)$ be a continuous derivation. There exists $h\in\mathscr{O}(C)$ such that $d=h\,d/dT$. Denote by $\ell_d:\Omega^1(C)\to\mathscr{O}(C)$ the $\mathscr{O}(C)$-linear map obtained by composing ℓ with the multiplication by h. We set

$$\nabla(d)\;:=\;(\ell_d\otimes 1)\circ\nabla\,. \tag{2.30}$$

The map $\nabla(d):\mathscr{F}(C)\to\mathscr{F}(C)$ provides $\mathscr{F}(C)$ with a structure of $(\mathscr{O}(C),d)$-differential module. Note that, for every $g\in\mathscr{O}(C)$, we have $\nabla(gd)=g\nabla(d)$. If the derivation d is clear, we often drop the symbol d and write ∇ instead of $\nabla(d)$.

From now on, we assume that $\mathscr{F}$ is free of finite rank r. We identify $\mathscr{F}(C)$ with $\mathscr{O}(C)^r$ and endow $\mathscr{F}(C)$ with a structure of normoid Fréchet space accordingly.

There exists a matrix $G(T) \in M_r(\mathscr{O}(C))$ such that the map $\nabla(d)$ is given by

$$d - G(T) \ : \ \mathscr{O}(C)^r \to \mathscr{O}(C)^r \ . \tag{2.31}$$

We denote by $\mathscr{O}(C)\langle d/dT\rangle$ the ring of differential polynomials with coefficients in $\mathscr{O}(C)$: the elements of $\mathscr{O}(C)\langle d/dT\rangle$ are abstract sums $\sum_{i=0}^n f_i \circ (d/dT)^i$ (so that we have an isomorphism of abelian groups $\mathscr{O}(C)\langle d/dT\rangle = \bigoplus_{i\geqslant 0} \mathscr{O}(C)\cdot (d/dT)^i$) and the multiplication law $\circ$ is the unique one satisfying the properties $f \circ (d/dT) = (d/dT)\circ f + df/dT$, for all $f \in \mathscr{O}(C)$.

Proposition 2.2.10. *Let f be an endomorphism of $\mathscr{O}(C)^r$ given by the multiplication by an $r\times r$ matrix with coefficients in $\mathscr{O}(C)\langle d/dT\rangle$. Then, f is a bounded endomorphism of $\mathscr{O}(C)^r$ that satisfies the compactness property of Definition 2.1.10.*

In particular, for every bounded derivation d, the endomorphism $\nabla(d)$ of $\mathscr{F}(C)$ defined above satisfies the compactness property of Definition 2.1.10.

Proof. We may extend the scalars in order to assume that K is not trivially valued and maximally complete.

The result now follows from [CM00, Proposition 8.2-2] (see also Remark 8.9-10 at the end of Section 8.2 of *ibid.*). □

Definition 2.2.11. *Let $b \in \{b_0, b_1\}$. We say that $(\mathscr{F},\nabla)$ is Fredholm at b if $\nabla(d/dT) : \mathscr{F}(C) \to \mathscr{F}(C)$ is. In this case, we define the* absolute generalized index *of $(\mathscr{F},\nabla)$ at b as*

$$\chi_b^{\mathrm{abs}}(\mathscr{F},\nabla) \ := \ \chi_b^{\mathrm{gen}}(\mathscr{F}(C),\nabla(d/dT)) - \chi_b^{\mathrm{gen}}(\mathscr{O}(C)^r, d/dT) \ . \tag{2.32}$$

We often remove ∇ from the notation when it is clear from the context.

The claim below shows that, in this definition, we could have chosen any other bounded derivation instead of d/dT, as soon as it is Fredholm at the boundary of the annulus.

Lemma 2.2.12. *Let $b \in \{b_0, b_1\}$. Let $d : \mathscr{O}(C) \to \mathscr{O}(C)$ be a bounded derivation that is Fredholm at b. Then $(\mathscr{F},\nabla)$ is Fredholm at b if, and only if, $\nabla(d) : \mathscr{F}(C) \to \mathscr{F}(C)$ is. Moreover, in this case, we have*

$$\chi_b^{\mathrm{abs}}(\mathscr{F},\nabla) \ := \ \chi_b^{\mathrm{gen}}(\mathscr{F}(C),\nabla(d)) - \chi_b^{\mathrm{gen}}(\mathscr{O}(C)^r, d) \ . \tag{2.33}$$

Proof. Let $d : \mathscr{O}(C) \to \mathscr{O}(C)$ be a bounded derivation. There exists $h \in \mathscr{O}(C)$ such that $d = h\, d/dT$. By Lemma 2.1.11 and Proposition 2.2.10, since d is Fredholm at b, so is h. Moreover, we have

$$\chi_b^{\mathrm{gen}}(\mathscr{O}(C)^r, hd/dT) \ = \ \chi_b^{\mathrm{gen}}(\mathscr{O}(C)^r, h) + \chi_b^{\mathrm{gen}}(\mathscr{O}(C)^r, d/dT) \ . \tag{2.34}$$

Similarly, $\nabla(hd/dT) = h\nabla(d/dT)$ is Fredholm at b if, and only if, $\nabla(d/dT)$ is and, in this case, we have

$$\chi_b^{\mathrm{gen}}(\mathscr{F}(C),\nabla(hd/dT)) \ = \ \chi_b^{\mathrm{gen}}(\mathscr{F}(C),h) + \chi_b^{\mathrm{gen}}(\mathscr{F}(C),\nabla(d/dT)) \ . \tag{2.35}$$

Finally, by Lemma 2.2.7, one has $\chi_b^{\mathrm{gen}}(\mathscr{F}(C),h) = \chi_b^{\mathrm{gen}}(\mathscr{O}(C)^r,h)$ and the result follows. □

Remark 2.2.13. *Let $b \in \{b_0, b_1\}$. If $(\mathscr{F},\nabla)$ is Fredholm at b, then, since $\chi_b^{\mathrm{gen}}(\mathscr{O}(C), T\frac{d}{dT}) = 0$, one has*

$$\chi_b^{\mathrm{abs}}(\mathscr{F},\nabla) \ = \ \chi_b^{\mathrm{gen}}(\mathscr{F}(C),\nabla(T\frac{d}{dT})) \ . \tag{2.36}$$

In particular, the absolute generalized index has all the properties of the usual generalized index. This is in fact the choice of derivation of [CM00].

2.2.3. Push-forward by standard ramification of irregularities and absolute indexes. In this section, we study the behavior of *generalized indexes* and *irregularities* of a connection under push-forward by *standard ramification*, *i.e.* by a finite étale morphism of the form by $T \mapsto T^n$.

Let $C = \{r_1 < |T| < r_2\}$, with $0 \leqslant r_1 < r_2 \leqslant +\infty$, be a standard open pseudo-annulus. Let $n \geqslant 1$. We denote by φ_n the endomorphism of $\mathbb{A}^{1,\mathrm{an}}_K$ that raises to the n^{th} power. It induces a finite étale morphism between C and the standard open pseudo-annuli

$$C^n := \varphi_n(C) = \{r_1^n < |T| < r_2^n\}\;. \tag{2.37}$$

Let

$$\varphi_n^\sharp \;:\; \mathscr{O}(C^n) \longrightarrow \mathscr{O}(C) \tag{2.38}$$

be the induced map on the rings of functions. For clarity, we will denote by $\widetilde{T}$ and T the coordinate functions on C and C^n respectively. For every $f(T) \in \mathscr{O}(C^n)$, we have $\varphi_n^\sharp(f(T)) = f(\widetilde{T}^n)$.

Proposition 2.2.14 ([Pul15, Section 5.3])**.** *Let b be a germ of segment in Γ_C. Denote by b^n its image in Γ_{C^n}. Let $\mathscr{F}$ be a differential equation over C. Then $\mathscr{F}$ has log affine radii along b if, and only if, $(\varphi_n)_*\mathscr{F}$ has log-affine radii along b^n. In this case, one has*

$$\mathrm{Irr}_{b^n}((\varphi_n)_*\mathscr{F}) \;=\; \mathrm{Irr}_b(\mathscr{F})\;. \tag{2.39}$$

Proof. A factorization of n into prime numbers corresponds to a factorization of the corresponding morphisms φ_n, therefore we can assume that n is a prime number. Let p be the characteristic of the residual field $\widetilde{K}$ of K. If $p = 0$, or if n is prime to p, the claim follows from [PP15, Lemmas 3.22 and 3.23]. If $n = p > 0$, the claim follows from [Ked10, Chapter 10] (cf. also [Pul15, Section 3]). □

We now focus on the absolute index. In the following, we use notation (2.30).

Corollary 2.2.15. *Let b be a germ of segment in Γ_C. Denote by b^n its image in Γ_{C^n}. Let $\mathscr{F}$ be a differential equation over C. Then $\mathscr{F}$ has finite generalized index at b if, and only if, its push-forward $(\varphi_n)_*\mathscr{F}$ has finite generalized index at b^n. In this case, we have*

$$\chi^{\mathrm{abs}}_{b^n}((\varphi_n)_*\mathscr{F}) \;=\; \chi^{\mathrm{abs}}_b(\mathscr{F})\;. \tag{2.40}$$

Proof. A simple computation shows that, for every $h(T) \in \mathscr{O}(C^n)$, we have

$$\Big(\frac{h(\widetilde{T}^n)}{n\widetilde{T}^{n-1}} \cdot \frac{d}{d\widetilde{T}}\Big) \circ \varphi_n^\sharp \;=\; \varphi_n^\sharp \circ \Big(h(T)\frac{d}{dT}\Big)\;. \tag{2.41}$$

We set $d := T\frac{d}{dT}$ and $\widetilde{d} := \frac{\widetilde{T}}{n} \cdot \frac{d}{d\widetilde{T}}$. It follows from (2.41) that the push-forward $(\varphi_n)_*\mathscr{F}$ is nothing but $\mathscr{F}$ seen as an $\mathscr{O}(C^n)$-module via $\varphi_n^\sharp$ and endowed with the connection

$$((\varphi_n)_*\nabla)(d) \;=\; \nabla(\widetilde{d})\;. \tag{2.42}$$

By Remark 2.2.13, we have $\chi^{\mathrm{abs}}_b(\mathscr{F}) = \chi^{\mathrm{gen}}_b(\nabla(\widetilde{d}))$ and $\chi^{\mathrm{abs}}_{b^n}((\varphi_n)_*(\mathscr{F})) = \chi^{\mathrm{gen}}_{b^n}(\nabla(\widetilde{d}))$. We now compute $\chi^{\mathrm{gen}}_b(\nabla(\widetilde{d}))$ and $\chi^{\mathrm{gen}}_{b^n}(\nabla(\widetilde{d}))$ as in Definition 2.2.3. We have

$$\mathscr{O}(C) \;=\; \bigoplus_{i=0}^{n-1} \widetilde{T}^i \varphi_n^\sharp(\mathscr{O}(C^n))\;. \tag{2.43}$$

The same is true for the decompositions

$$\widetilde{T}^k\mathscr{O}(D_0(C)) \;=\; \bigoplus_{i=0}^{n-1} \widetilde{T}^{i+k}\varphi_n^\sharp(\mathscr{O}(D_0(C^n))) \quad \text{and} \quad \widetilde{T}^m\mathscr{O}(D_1(C)) \;=\; \bigoplus_{i=0}^{n-1} \widetilde{T}^{m+i}\varphi_n^\sharp(\mathscr{O}(D_1(C^n)))\;. \tag{2.44}$$

It follows that the truncations (2.3) that we consider in Definition 2.2.3 before and after push-forward actually are the same. In particular, the generalized indexes, at b and b^n, are equal before and after push-forward. □

2.2.4. Absolute generalized indexes for connections over general open pseudo-annuli. Let C be an open pseudo-annulus. Denote by b_0 and b_1 the germs of segments at the open boundary of C.

Let Ω be a spherically complete and algebraically closed field extension of K such that $|\Omega| = \mathbb{R}_{\geqslant 0}$. By [Liu87, Proposition 3.2]), C_Ω may be identified with an analytic domain of $\mathbb{P}^{1,\mathrm{an}}_\Omega$. It is hence a finite disjoint union $C_\Omega = C_1 \sqcup \ldots \sqcup C_n$ of standard open pseudo-annuli over Ω (see Definition 2.2.1).

Let $i \in \{1, \ldots, n\}$. For every $k \in \{0, 1\}$, there exists a unique germ of segment in C_i over b_k. We denote it by $b_{k,i}$. The open boundary of C_i contains exactly the two germs of segments $b_{0,i}$ and $b_{1,i}$.

Let $(\mathscr{F}, \nabla)$ be a differential equation on C. The space C_Ω being the disjoint union of the C_i's, we have a direct sum decomposition of Ω-vector spaces

$$(\mathscr{F}_\Omega)_{|C} \;=\; \bigoplus_{i=1}^{n} \mathscr{F}_i \;, \tag{2.45}$$

where $\mathscr{F}_i := (\mathscr{F}_\Omega)_{|C_i}$. For every $i \in \{1, \ldots, n\}$, denote by ∇_i the connection on $\mathscr{F}_i$ induced by ∇. Since K is maximally complete, by [Laz62], $\mathscr{F}_i$ is a free $\mathscr{O}_{C_i}$-module.

We fix a coordinate T_1 on C_1 with respect to which we will compute the generalized indexes at $b_{0,1}$ and $b_{1,1}$ (see Section 2.2.1). By [PP15, Corollary 2.20], for every $i \in \{2, \ldots, n\}$, there exists a continuous K-linear automorphism of Ω that sends C_1 to C_i. We fix such an automorphism σ_i.The image of T_1 by σ_i is a coordinate T_i on C_i with respect to which we will compute the generalized indexes at $b_{0,i}$ and $b_{1,i}$.

With these choices, given $i, j \in \{1, \ldots, n\}$ and $k \in \{0, 1\}$, $\mathscr{F}_i$ is Fredholm at $b_{k,i}$ if, and only if, $\mathscr{F}_j$ is Fredholm at $b_{k,j}$ and, in this case, we have

$$\chi^{\mathrm{abs}}_{b_{k,i}}(\mathscr{F}_i) \;=\; \chi^{\mathrm{abs}}_{b_{k,j}}(\mathscr{F}_j) \;. \tag{2.46}$$

Definition 2.2.16. *Let $k \in \{0, 1\}$. We say that $(\mathscr{F}, \nabla)$ is Fredholm at b_k if, for some (or equivalently every) $i \in \{1, \ldots, n\}$, $\mathscr{F}_i$ is Fredholm at $b_{k,i}$. In this case, we define the absolute generalized index of $(\mathscr{F}, \nabla)$ at b_k as*

$$\chi^{\mathrm{abs}}_{b_k}(\mathscr{F}) \;:=\; \sum_{i=1}^{n} \chi^{\mathrm{abs}}_{b_{k,i}}(\mathscr{F}_i) \;. \tag{2.47}$$

Remark 2.2.17. *By Lemma 2.1.4, the property of being Fredholm at b_k and, in this case, the value of $\chi^{\mathrm{abs}}_{b_k}$ remain unchanged if we replace Ω by a bigger extension and use the coordinates induced by the T_i's. In particular, if C is a standard open pseudo-annulus and if $\mathscr{F}$ is free, then Definition 2.2.16 agrees with Definition 2.2.11 (for suitable coordinates).*

Lemma 2.2.18. *The following assertions are equivalent.*

i) $\mathscr{F}$ is Fredholm at b_0 and b_1;

ii) there exists a complete valued extension L of K with non-trivial valuation such that $\mathscr{F}_L$ has finite-dimensional de Rham cohomology over C_L;

iii) for every complete valued extension L of K, $\mathscr{F}_L$ has finite-dimensional de Rham cohomology over C_L.

Moreover, when these conditions are satisfied, for every complete valued extension L of K, we have

$$\chi_{\mathrm{dR}}(C_L, \mathscr{F}_L) \;=\; \chi^{\mathrm{abs}}_{b_0}(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_1}(\mathscr{F})\;. \tag{2.48}$$

Proof. By Lemma 2.2.5, Remark 2.2.13 and Proposition 2.2.10, $\mathscr{F}$ is Fredholm at b_0 and b_1 if, and only if, for some (or equivalently all) $i \in \{1, \dots, n\}$, $\mathscr{F}_i$ has finite-dimensional de Rham cohomology over C_i if, and only if, $\mathscr{F}_\Omega$ has finite-dimensional de Rham cohomology over C_Ω. Moreover, in this case, we have

$$\chi_{\mathrm{dR}}(C_\Omega, \mathscr{F}_\Omega) = \sum_{i=1}^{n} \chi_{\mathrm{dR}}(C_i, \mathscr{F}_i) \tag{2.49}$$

$$= \sum_{i=1}^{n} \chi^{\mathrm{abs}}_{b_{0,i}}(\mathscr{F}_i) + \chi^{\mathrm{abs}}_{b_{1,i}}(\mathscr{F}_i) \tag{2.50}$$

$$= \chi^{\mathrm{abs}}_{b_0}(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_1}(\mathscr{F}). \tag{2.51}$$

It follows that i) implies ii), that iii) implies i) and that (2.48) holds for $L = \Omega$.

By Theorem 1.4.7, ii) implies iii) and the value of $\chi_{\mathrm{dR}}(C_L, \mathscr{F}_L)$ is independent of L. The result follows. □

Remark 2.2.19. *Having developed the theory, we will actually be able to prove that, under natural assumptions on $(\mathscr{F}, \nabla)$, $\chi^{\mathrm{abs}}_{b_{k,i}}(\mathscr{F}, \nabla)$ is independent of the coordinate chosen on C_i (see Proposition 2.3.12).*

From now on, we will not mention the field Ω nor the coordinates T_i when speaking about being Fredholm at a germ of segment. Still, different choices may lead to different definitions and we advise the reader to use those results with care. We often use implicitly the natural choices: for instance, when we pass from a pseudo-annulus to a smaller one, we use the same coordinates.

2.3. Index formula over an open pseudo-annulus.

Let C be an open pseudo-annulus. Let $\mathscr{F}$ be a differential equation of rank r on C. Recall that, by [PP13, Corollary 6.2.28], if all the radii of $\mathscr{F}$ are log-affine radii along Γ_C, then all those radii are locally constant outside Γ_C. In particular, the radii are separated over C if, and only if, they are separated along Γ_C.

Definition 2.3.1 (Robba property)**.** *We say that an index $i \in \{1, \dots, r\}$ satisfies the* Robba property *if $\mathcal{R}_i(x, \mathscr{F}) = 1$ for all x in the skeleton Γ_C of C.*

Definition 2.3.2. *Assume that all the radii of $\mathscr{F}$ are log-affine along Γ_C. If some index satisfies the Robba property, we denote by i_R the smallest that does. By [PP13, Theorem 5.3.1], there exists a unique sub-object $\mathscr{F}^{\mathrm{Robba}}$ of $\mathscr{F}$ of rank $r - i_R + 1$ all of whose indexes satisfy the Robba property.*

If none of the indexes satisfy the Robba property, we set $i_R := r + 1$ and $\mathscr{F}^{\mathrm{Robba}} := 0$.

Remark 2.3.3. *In the setting above the quotient $\mathscr{F}/\mathscr{F}^{\mathrm{Robba}}$ is a differential equation of rank $i_R - 1$ and, for each $x \in C$ and $j \in \{1, \dots, i_R - 1\}$, we have*

$$\mathcal{R}_j(x, \mathscr{F}/\mathscr{F}^{\mathrm{Robba}}) = \mathcal{R}_j(x, \mathscr{F}) < 1\;. \tag{2.52}$$

This follows from [PP13, Theorem 5.3.1] in the first case and it is obvious in the second.

Remark 2.3.4. *If the radii of $\mathscr{F}$ are not log-affine along Γ_C, it is not clear whether an analogue of $\mathscr{F}^{\mathrm{Robba}}$ exists.*

Proposition 2.3.5. *Assume that all the radii of $\mathscr{F}$ are log-affine along Γ_C. Then $\mathscr{F}$ has finite index if, and only if, $\mathscr{F}^{\mathrm{Robba}}$ has. Moreover, in this case, for $i = 0, 1$, we have*

$$\mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}^{\mathrm{Robba}}) = \mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}) , \quad \chi_{\mathrm{dR}}(C, \mathscr{F}^{\mathrm{Robba}}) = \chi_{\mathrm{dR}}(C, \mathscr{F}) . \tag{2.53}$$

Proof. By Proposition 1.4.9, Situation 1, for all i, we have $\mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F}/\mathscr{F}^{\mathrm{Robba}}) = 0$. The result now follows by writing the cohomology long exact sequence associated to the short exact sequence $0 \to \mathscr{F}^{\mathrm{Robba}} \to \mathscr{F} \to \mathscr{F}/\mathscr{F}^{\mathrm{Robba}} \to 0$. □

2.3.1. Absolute generalized index and irregularity.

Lemma 2.3.6. *Let C be an open pseudo-annulus and let $\mathscr{F}$ be a differential equation on C. Let C' be an open sub-pseudo-annulus of C with $\Gamma_{C'} \subseteq \Gamma_C$. Assume that there exists a complete valued extension L of K with non-trivial valuation such that the equations $\mathscr{F}_L$ and $(\mathscr{F}_L)_{|C'_L}$ have finite-dimensional de Rham cohomology and $\chi_{\mathrm{dR}}(C_L, \mathscr{F}_L) = \chi_{\mathrm{dR}}(C'_L, (\mathscr{F}_L)_{|C'_L})$. Then, for each $i \in \{0, 1\}$, the restriction map*

$$\mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(C', \mathscr{F}_{|C'}) \tag{2.54}$$

is an isomorphism.

Proof. For $i = 0$ the restriction map (2.54) is injective, while for $i = 1$ it is surjective by Lemma 1.4.8. By Theorem 1.4.7, we have $\chi_{\mathrm{dR}}(C, \mathscr{F}) = \chi_{\mathrm{dR}}(C', \mathscr{F}_{|C'})$ and the result follows. □

Remark 2.3.7. *Theorem 2.3.14 will provide conditions that ensure that the assumptions of Lemma 2.3.6 are satisfied.*

We state here a result that is related to Lemma 2.3.6 and that will be useful later on.

Lemma 2.3.8. *Let X be a quasi-smooth K-analytic curve. Let Z be a locally finite subset of rigid points of X and let $\mathcal{F}$ be a differential equation on X with meromorphic singularities on Z. Let X' be an analytic domain of X and let $C := \bigsqcup_i C_i$ be a disjoint union of open pseudo-annuli of X. Assume that*

i) $X' \cup C = X$;

ii) $Z \cap C = \emptyset$;

iii) for each i, $C'_i := X' \cap C_i$ is an open pseudo-annulus in C_i such that $\Gamma_{C'_i} \subseteq \Gamma_{C_i}$;

iv) for each i and each $k = 0, 1$, the restriction map $\mathrm{H}^k_{\mathrm{dR}}(C_i, \mathcal{F}_{|C_i}) \to \mathrm{H}^k_{\mathrm{dR}}(C'_i, \mathcal{F}_{|C'_i})$ is an isomorphism.

Then, for each $k \geqslant 0$, the restriction map

$$\mathrm{H}^k_{\mathrm{dR}}(X(*Z), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^k_{\mathrm{dR}}(X'(*Z), \mathcal{F}_{|X'}) \tag{2.55}$$

is an isomorphism.

Proof. Let $C' := \bigsqcup_i C'_i$. Then $X = X' \cup C$ and $C \cap X' = C'$. By ii) we have $\mathrm{H}^k_{\mathrm{dR}}(C(*Z), \mathcal{F}_{|C}) = \mathrm{H}^k_{\mathrm{dR}}(C, \mathcal{F}_{|C})$ and $\mathrm{H}^k_{\mathrm{dR}}(C'(*Z), \mathcal{F}_{|C'}) = \mathrm{H}^k_{\mathrm{dR}}(C', \mathcal{F}_{|C'})$. The Mayer-Vietoris sequence then gives

$$\cdots \to \mathrm{H}^{i-1}_{\mathrm{dR}}(C', \mathcal{F}_{|C'}) \to \mathrm{H}^i_{\mathrm{dR}}(X(*Z), \mathcal{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X'(*Z), \mathcal{F}_{|X'}) \oplus \mathrm{H}^i_{\mathrm{dR}}(C, \mathcal{F}_{|C}) \to \mathrm{H}^i_{\mathrm{dR}}(C', \mathcal{F}_{|C'}) \to \cdots \tag{2.56}$$

By iv) the restriction map $\mathrm{H}^i_{\mathrm{dR}}(C, \mathcal{F}_{|C}) \to \mathrm{H}^i_{\mathrm{dR}}(C', \mathcal{F}_{|C'})$ is an isomorphism. The claim follows. □

Lemma 2.3.9. *Let C be an open pseudo-annulus and let $\mathscr{F}$ be a differential equation on C. Assume that there exists a complete valued extension L of K with non-trivial valuation such that, for every open pseudo-annulus $C' \subseteq C$ with $\Gamma_{C'} \subseteq \Gamma_C$, the equation $(\mathscr{F}_L)_{|C'_L}$ has finite-dimensional de Rham cohomology over C'_L and $\chi_{\mathrm{dR}}(C'_L, (\mathscr{F}_L)_{|C'_L}) = 0$.*

Then, for every germs of segments b and b' in Γ_C, every open sub-pseudo-annuli C_b and $C_{b'}$ of C whose open boundaries contains b and b' respectively, $\mathscr{F}_{|C_b}$ and $\mathscr{F}_{|C_{b'}}$ are Fredholm at b and b' respectively and we have

$$\chi_b^{\mathrm{abs}}(\mathscr{F}_{|C_b}) \;=\; \varepsilon(b,b')\,\chi_{b'}^{\mathrm{abs}}(\mathscr{F}_{|C_{b'}})\;, \tag{2.57}$$

where $\varepsilon(b,b')$ is equal to 1 (resp. -1) if b and b' have the same orientation (resp. opposite orientations).

Proof. Let b and b' be two germs of segment in Γ_C and let C_b and $C_{b'}$ be as in the statement.

Let b_0 (resp. b_1) be the germ of segment at the open boundary of C whose orientation is opposite to (resp. equal to) that of b. There exist an open sub-pseudo-annulus C'_b of C whose open boundary is $\{b_0, b\}$. Note that we necessarily have $\Gamma_{C'_b} \subseteq \Gamma_C$. By Lemma 2.2.18 and Lemma 2.2.4, $\mathscr{F}_{|C'_b}$ is Fredholm at b and we have

$$\chi_b^{\mathrm{abs}}(\mathscr{F}_{|C'_b}) = -\chi_{b_0}^{\mathrm{abs}}(\mathscr{F}_{|C'_b}) = \chi_{b_0}^{\mathrm{abs}}(\mathscr{F})\;. \tag{2.58}$$

By Lemma 2.2.4, the same results hold with C'_b replaced by C_b.

Similarly, we show that $\mathscr{F}_{|C_{b'}}$ is Fredholm at b' and that $\chi_{b'}^{\mathrm{abs}}(\mathscr{F}_{|C_{b'}})$ is equal to $\chi_{b_0}^{\mathrm{abs}}(\mathscr{F})$ if b' has the same orientation as b or to $\chi_{b_1}^{\mathrm{abs}}(\mathscr{F})$ if b' has the opposite orientation. By Lemma 2.2.18 again, we have

$$\chi_{b_0}^{\mathrm{abs}}(\mathscr{F}) + \chi_{b_1}^{\mathrm{abs}}(\mathscr{F}) = 0 \tag{2.59}$$

and the result follows. □

Proposition 2.3.10. *Let C be an open pseudo-annulus, let b be a germ of segment at the open boundary of C and let $\mathscr{F}$ be a differential equation on C with log-affine radii along Γ_C.*

Let

$$\mathscr{F} \;=\; \mathscr{F}^{\mathrm{Robba}} \oplus \mathscr{F}^{<\mathrm{sol}} \tag{2.60}$$

be the decomposition of $\mathscr{F}$ into its Robba part and its spectral non-solvable part. Then $\mathscr{F}^{<\mathrm{sol}}$ is Fredholm at b and one has

$$\chi_b^{\mathrm{abs}}(\mathscr{F}^{<\mathrm{sol}}) \;=\; \mathrm{Irr}_b(\mathscr{F}^{<\mathrm{sol}}) \;=\; \mathrm{Irr}_b(\mathscr{F})\;. \tag{2.61}$$

In particular, the following conditions are equivalent:

i) $\mathscr{F}^{\mathrm{Robba}}$ is Fredholm at b and

$$\chi_b^{\mathrm{abs}}(\mathscr{F}^{\mathrm{Robba}}) \;=\; 0\;; \tag{2.62}$$

ii) $\mathscr{F}$ is Fredholm at b and

$$\chi_b^{\mathrm{abs}}(\mathscr{F}) \;=\; \mathrm{Irr}_b(\mathscr{F})\;. \tag{2.63}$$

Proof. By Definition 2.2.16, we may assume that K is algebraically closed, spherically complete with $|K| = \mathbb{R}_{\geqslant 0}$ and that C is a standard open pseudo-annulus $C = \{r_1 < |T| < r_2\}$ with $0 \leqslant r_1 < r_2 \leqslant +\infty$. In this case, the equivalence of i) and ii) follows from (2.61), by additivity of generalized index and irregularity.

We are then reduced to prove (2.61), therefore we can assume that $\mathscr{F} = \mathscr{F}^{<\mathrm{sol}}$.

Let us first assume that the residual field of K has positive characteristic $p > 0$. Thanks to Proposition 1.4.9, Lemma 2.3.9 applies, so we can shrink C and assume that there exists $r < 1$ such that all the radii of $\mathscr{F}$ are smaller than r at all points of Γ_C (*i.e.* the radii are not approaching 1 at the open boundary of C). This implies that the push-forward by the Frobenius morphism φ_p reduces the radii uniformly on Γ_C, hence that there exists $n > 0$ such that all the radii of $(\varphi_p)^n_*(\mathscr{F})$ are all strictly smaller than the radius of Young's disk [You92] at each point of Γ_C. Corollary 2.2.15 and Proposition 2.2.14 show that the irregularity and the generalized index behave in the same way by Frobenius push-forward.

Replacing $\mathscr{F}$ by $(\varphi_p)^n_*(\mathscr{F})$, we can assume that the radii of $\mathscr{F}$ are all strictly smaller than the radius of Young's disk [You92] at each point of Γ_C. Shrinking C again and decomposing $\mathscr{F}$ by the radii, we can assume that all the radii of $\mathscr{F}$ are equal to the same log-affine function along Γ_C.

Since the determinant of Katz's base change matrix (see [Kat87b]) has a finite number of zeros on every closed annulus (this is independent on the residual characteristic), by restricting C again, we can assume that $\mathscr{F}$ admits a cyclic basis over C.

For these reasons we can assume that

a) $\mathscr{F}$ is a cyclic module;

b) the radii $\mathcal{R}_i(-,\mathscr{F})$ of $\mathscr{F}$ are all equal to the same log-affine function on Γ_C;

c) the radii $\mathcal{R}_i(-,\mathscr{F})$ of $\mathscr{F}$ are all strictly smaller than Young's bound along Γ_C.

If the residual field of K has characteristic $p = 0$, we can again assume that a), b), c) hold. The proof is essentially the same, with the difference that assumption c) is automatically verified (i.e. Frobenius push-forward in not needed). Indeed, if $p = 0$, Young's bound is 1, so the radii are automatically all strictly smaller than it because $\mathscr{F} = \mathscr{F}^{<\mathrm{sol}}$.

Under a), b), c), the irregularity and the generalized index are explicitly intelligible in a cyclic basis. From now on the proof is the same in all residual characteristics.

Set $d := \frac{d}{dT}$, and denote by x_ρ the sup-norm on the annulus $\{|T| = \rho\}$. We recall that the norm of d as an operator on $\mathscr{O}(C)$ endowed with the norm x_ρ is given by

$$|d|(x_\rho) \;=\; \sup_{f\in\mathscr{O}(C)-\{0\}} \frac{|d(f)|(x_\rho)}{|f|(x_\rho)} \;=\; \rho^{-1} \;=\; |T|^{-1}\,. \tag{2.64}$$

Consider a cyclic basis $(v, \nabla(v), \nabla^2(v), \ldots, \nabla^{r-1}(v))$ of $\mathscr{F}$. In such a basis the differential equation $\mathscr{F}$ is associated to an operator

$$P(T,d) \;=\; \sum_{k=0}^{r} a_k(T) d^k\,, \tag{2.65}$$

where $a_r = 1$. More precisely we have $\nabla = d - G(T)$, where $G(T)$ is the companion matrix of the polynomial $P(T,d)$:

$$G(T) \;=\; \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & Id & \\ 0 & & & \\ \hline -a_0 & -a_1 & \cdots & -a_{r-1} \end{array}\right)\,. \tag{2.66}$$

Since the radii are all small, the convergence Newton polygon of $\mathscr{F}$ coincides with the spectral Newton polygon of $P(T,d)$ (cf. [Pul15, Proposition 4.11] or also [CM02, Théorème 6.2]).[2]

Since the radii are all equal to the same log-affine function, the Young's spectral Newton polygon associated to $P(T,d)$ has no breaks, hence the radii are directly related to the norm of the constant

[2] Recall that the convergence Newton polygon is the polygon whose slopes are $\{\log(\mathcal{R}_i(x_\rho,\mathscr{F}))\}_{i=1,\ldots,r}$, while the spectral Newton polygon of $P(T,d)$ is defined by considering the convex hull of the x_ρ-values of the coefficients of $P(T,d)$, cf. [Pul15, Proposition 4.11].

term $a_0(T) \in \mathscr{O}(C)$ (note that $a_0(T)$ is not zero since, by construction, $\mathscr{F}$ has no trivial submodules). More precisely, Young's theorem ensures that for all $k = 1, \dots, r-1$ and all $\rho \in]r_1, r_2[$, one has

$$|a_0|(x_\rho)^{1/r} \;\; > \;\; \max(\, |d|(x_\rho)\;,\; |a_k|(x_\rho)^{\frac{1}{r-k}}\;)\;, \tag{2.67}$$

and that for all $i = 1, \dots, r$

$$\mathcal{R}_i(x_\rho, \mathscr{F}) \;=\; \omega \cdot |a_0|(x_\rho)^{-\frac{1}{r}} \rho^{-1}\;,[3] \tag{2.68}$$

where ω is either 1 or p if the characteristic of the residual field $\widetilde{K}$ is 0 or $p > 0$ respectively (cf. (0.1)). In particular, since the radii are assumed to be log-affine along Γ_C, a_0 has no zeros in C. We also deduce that

$$\mathrm{Irr}_b(\mathscr{F}) \;=\; -r\partial_b|T| - \partial_b\,|a_0(T)|\;. \tag{2.69}$$

We now compute the absolute index. First of all, by Lemma 2.3.9, we can replace b by another germ of segment in Γ_C with the same orientation. We then pick $\gamma \in]r_1, r_2[$ and assume that b is a germ of segment out of x_γ.

Let $\beta \in K$ such that

$$|\beta| \;=\; |a_0|(x_\gamma)^{1/r}\;. \tag{2.70}$$

We now replace the cyclic basis of $\mathscr{F}$ by the following more convenient one

$$(v, \beta^{-1}\nabla(v), \beta^{-2}\nabla^2(v), \dots, \beta^{1-r}\nabla^{r-1}(v))\;. \tag{2.71}$$

In this basis, the connection is given by $d - G_\beta(T)$ where

$$G_\beta(T) \;=\; \beta \cdot \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & Id & \\ 0 & & & \\ \hline -\beta^{-r}a_0 & -\beta^{-r+1}a_1 & \cdots & -\beta^{-1}a_{r-1} \end{array}\right)\;, \tag{2.72}$$

and its inverse is given by

$$G_\beta(T)^{-1} \;=\; \beta^{-1} \cdot \left(\begin{array}{ccc|c} -\beta\frac{a_1}{a_0} & \cdots & -\beta^{r-1}\frac{a_{r-1}}{a_0} & -\beta^r\frac{1}{a_0} \\ \hline & & & 0 \\ & Id & & \vdots \\ & & & 0 \end{array}\right)\;. \tag{2.73}$$

The reason of these choices is that, in this situation, we have (cf. (2.67))

$$|G_\beta|(x_\gamma) \;=\; |\beta| \cdot \max(1, |\beta|^{-r}|a_0|(x_\gamma), |\beta|^{1-r}|a_1|(x_\gamma), \dots, \beta^{-1}|a_{r-1}|(x_\gamma)) \;=\; |\beta|\;, \tag{2.74}$$

$$|G_\beta^{-1}|(x_\gamma) \;=\; |\beta|^{-1} \cdot \max(1, |\beta|^r\frac{1}{|a_0|(x_\gamma)}, |\beta|^1\frac{|a_1|(x_\gamma)}{|a_0|(x_\gamma)}, \dots, \beta^{r-1}\frac{|a_{r-1}|(x_\gamma)}{|a_0|(x_\gamma)}) \;=\; |\beta|^{-1}\;. \tag{2.75}$$

By Proposition 1.4.9, $\mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{F})$ and $\mathrm{H}^1_{\mathrm{dR}}(C, \mathscr{F})$ are finite-dimensional. In other words, the endomorphism $Td - TG_\beta$ of $\mathscr{O}(C)^r$ is Fredholm, therefore, by Proposition 2.1.12, it is Fredholm at b. By Remark 2.2.13, we have $\chi_b^{\mathrm{gen}}(Td - TG_\beta) = \chi_b^{\mathrm{abs}}(\mathscr{F})$.

We now write

$$Td - TG_\beta \;=\; (Td(TG_\beta)^{-1} - 1) \circ (TG_\beta). \tag{2.76}$$

By Lemma 2.1.11, we have

$$\chi_b^{\mathrm{gen}}(Td - TG_\beta) \;=\; \chi_b^{\mathrm{gen}}(Td \circ (TG_\beta)^{-1} - 1) + \chi_b^{\mathrm{gen}}(TG_\beta)\;. \tag{2.77}$$

[3] The presence of ρ^{-1} in (2.68) is due to the fact that we consider normalized radii instead of spectral radii as in [Pul15, Proposition 4.11].

The generalized index of the last term is obtained from Lemma 2.2.6 and (2.69):

$$\chi_b^{\mathrm{gen}}(TG_\beta) \;=\; -\partial_b(\det(TG_\beta)) \;=\; -\partial_b|T^r\cdot\beta^{1-r}a_0(T)| \;=\; -r\partial_b|T|-\partial_b|a_0(T)| \;=\; \mathrm{Irr}_b(\mathscr{F})\;. \quad (2.78)$$

With the notations of Definition 2.2.3, we consider a sub-annulus of $C_b=\{r_1'<|T|<r_2'\}$ C having b at its open boundary and we fix $k\in\{0,1\}$ in order that $b=b_k$. To conclude the proof, we will show that the truncated operator $p_k\circ(Td\circ(TG_\beta)^{-1}-1)\circ i_k$ (cf. (2.3)) is invertible. This will imply that $\chi_b^{\mathrm{gen}}(Td\circ(TG_\beta)^{-1}-1)=0$.

Let D_b be the disk in $\mathbb{P}^{1,\mathrm{an}}_K$ having b at its open boundary. It contains C_b.

Let $\rho\in]r_1',r_2'[$. Endowing $\mathscr{O}(C_b)$ and $\mathscr{O}(D_b)$ with the norm x_ρ, the inclusion $i_k\colon \mathscr{O}(D_b)\to\mathscr{O}(C_b)$ is isometric and the projection $p_k\colon\mathscr{O}(C_b)\to\mathscr{O}(D_b)$ is a contraction. We deduce that

$$|p_k\circ(Td\circ(TG_\beta)^{-1})\circ i_k|(x_\rho)\leqslant|Td\circ(TG_\beta)^{-1}|(x_\rho)\;. \quad (2.79)$$

On the other hand, it follows from (2.64) and (2.75) that

$$|Td\circ(TG_\beta)^{-1}|(x_\gamma)\;\leqslant\;|Td|(x_\gamma)\cdot|(TG_\beta)^{-1}|(x_\gamma)\;=\;|(TG_\beta)^{-1}|(x_\gamma)\;=\;|(TG_\beta)|(x_\gamma)^{-1}\;<\;1\;, \quad (2.80)$$

where the last strict inequality follows from (2.64) and (2.67) .

By continuity, the strict inequality holds for all ρ sufficiently close to γ. Since the boundary of C_b in C contains x_γ, up to shrinking C_b, we can assume that it holds for all $\rho\in]r_1',r_2'[$. It follows that $|p_k\circ(Td\circ(TG_\beta)^{-1})\circ i_k|(x_\rho)<1$ for all $\rho\in]r_1',r_2'[$. We deduce that $p_k\circ(Td\circ(TG_\beta)^{-1}-1)\circ i_k=p_k\circ(Td\circ(TG_\beta)^{-1})\circ i_k-1$ is invertible. The claim follows. □

Corollary 2.3.11. *Let D be an open pseudo-disk inside X (cf. [PP13, Definition 1.1.8]). Let b_D be the germ of segment at infinity on D (oriented towards the interior of D). Let $\mathscr{F}$ be a differential equation over D that is Fredholm at b_D and whose radii are all log-affine along b_D. Let C be an open pseudo-annulus in D containing b_D such that the radii of $\mathscr{F}$ are log-affine on Γ_C.*

Then $\mathscr{F}$ has finite-dimensional de Rham cohomology over D and we have

$$\begin{aligned}\chi_{\mathrm{dR}}(D,\mathscr{F}) \;&=\; \chi_{b_D}^{\mathrm{abs}}(\mathscr{F})+r\;, &(2.81)\\ &=\;\chi_{b_D}^{\mathrm{abs}}(\mathscr{F}_{|C}^{\mathrm{Robba}})+\mathrm{Irr}_{b_D}(\mathscr{F})+r\;, &(2.82)\\ &=\;\chi_{b_D}^{\mathrm{abs}}(\mathscr{F}_{|C}^{\mathrm{Robba}})+\mathrm{h}^0_{\mathrm{dR}}(D,\mathscr{F})-\partial_{b_D}H_r(-,\mathscr{F})^{4}\;, &(2.83)\end{aligned}$$

where $r:=\mathrm{rank}(\mathscr{F}_{|D})$. In particular, if $\chi_{b_D}^{\mathrm{abs}}(\mathscr{F}_{|C}^{\mathrm{Robba}})=0$, we obtain

$$\mathrm{h}^1_{\mathrm{dR}}(D,\mathscr{F})\;=\;\partial_{b_D}H_r(-,\mathscr{F})\;. \quad (2.84)$$

If, moreover, the radii of $\mathscr{F}$ are spectral non-solvable on b_D, then we have $\chi_{\mathrm{dR}}(D,\mathscr{F})\leqslant 0$ and $\chi_{\mathrm{dR}}(D,\mathscr{F})=0$ if, and only if, $\partial_{b_D}H_r(-,\mathscr{F})=0$.

Proof. All the quantities are stable by scalar extension. Therefore we can assume that K is spherically complete, algebraically closed, and that $|K|=\mathbb{R}_{\geqslant 0}$, that D is a disk defined by $D=\{|T|<r_2\}$ and that $C_b=\{r_1<|T|<r_2\}$. We are then in the context of Definition 2.2.3, which we now apply with $D=D_0$, $n=0$, and $m=-1$. The truncation $p_0\circ\nabla(d/dT)\circ i_0$ (cf. (2.3)) of the connection $\nabla(d/dT):\mathscr{F}(C)\to\mathscr{F}(C)$ coincides with our original connection $\nabla(d/dT):\mathscr{F}(D)\to\mathscr{F}(D)$. Hence

[4] Note that the slope of the total height of the convergence Newton polygon $\partial_{b_D}H_r(-,\mathscr{F})$ is taken before localization to C.

$\chi^{\mathrm{gen}}_{b_D}(\mathscr{F}(C),\nabla(d/dT)) = \chi(\mathscr{F}(D),\nabla(d/dT)) = \chi_{\mathrm{dR}}(D,\mathscr{F})$. We deduce that

$$\chi^{\mathrm{abs}}_{b_D}(\mathscr{F}) = \chi^{\mathrm{gen}}_{b_D}(\mathscr{F}(C),\nabla(d/dT)) - \chi^{\mathrm{gen}}_{b_D}(\mathscr{O}(C),d/dT) \tag{2.85}$$

$$= \chi_{\mathrm{dR}}(D,\mathscr{F}) - \chi_{\mathrm{dR}}(D,\mathscr{O}) \tag{2.86}$$

$$= \chi_{\mathrm{dR}}(D,\mathscr{F}) - r\;, \tag{2.87}$$

hence (2.81) holds. Formulas (2.82), (2.83) and (2.84) now follow immediately from Proposition 2.3.10 and [PP13, Lemma 2.8.4].

If the radii are all spectral non solvable at b_D, we have $\mathrm{H}^0_{\mathrm{dR}}(D,\mathscr{F}) = 0$. Indeed any global solution of $\mathscr{F}$ on D generates an over-solvable radius along b_D. Hence $\chi_{\mathrm{dR}}(D,\mathscr{F}) = -\mathrm{h}^1_{\mathrm{dR}}(D,\mathscr{F}) \leqslant 0$. The last assertion then follows from (2.84). □

2.3.2. Invariance of the absolute index. We now prove that the absolute index is an intrinsic notion. The invariance actually follows from Proposition 2.3.10 in the spectral non solvable case, and from Proposition A.1.5 in the solvable case.

Proposition 2.3.12 (Invariance of the absolute index)**.** *Let C be an open pseudo-annulus and let b be a germ of segment at the open boundary of C. Let $\mathscr{F}$ be a finite differential equation over C whose radii are all log-affine along b.*

Let $\sigma : C \xrightarrow{\sim} C$ be an automorphism of C. Then

$$\chi^{\mathrm{abs}}_{b}(\mathscr{F}) \;=\; \chi^{\mathrm{abs}}_{\sigma^{-1}(b)}(\sigma^*\mathscr{F})\;. \tag{2.88}$$

In particular, the absolute generalized index $\chi^{\mathrm{abs}}_b(\mathscr{F})$ is independent of Ω and of the coordinates T_i (see Section 2.2.4).

Proof. By Definition 2.2.16 we may assume that K is algebraically closed, spherically complete with $|K| = \mathbb{R}_{\geqslant 0}$ and that C is a standard open pseudo-annulus $C = \{r_1 < |T| < r_2\}$ with $0 \leqslant r_1 < r_2 \leqslant +\infty$. An inversion of the orientation is possible only if $(r_1,r_2) = (0,\infty)$ or $r_1,r_2 \notin \{0,\infty\}$. If $\sigma(b) \neq b$, we consider the automorphism ψ defined by

$$\psi(T) := \begin{cases} T^{-1} & \text{if} \quad r_1 = 0,\ r_2 = +\infty \\ a_1\cdot a_2\cdot T^{-1} & \text{if} \quad r_1,r_2 \in]0,+\infty[\end{cases} \tag{2.89}$$

where $a_i \in K$, $|a_i| = r_i$ for $i = 1,2$. Equality (2.88) is satisfied by ψ and ψ^{-1}, who only exchange the roles of D_0 and D_∞ in the definition of χ^{abs}_b. Now $\sigma' := \sigma\circ\psi^{-1}$ preserves the orientation. Since $\sigma = \sigma'\circ\psi$, it is enough to prove the assertion for σ'. In other words, we can assume that σ preserves the orientation of C, *i.e.* $\sigma(b) = b$.

Since $\sigma(T)$ is invertible in $\mathscr{O}(C)$, it can be written as

$$\sigma(T) \;=\; q(T^d + h(T)) \tag{2.90}$$

with $q \in K^*$ (and $|q| = 1$ if $(r_1,r_2) \neq (0,+\infty)$) and $h(T) \in \mathscr{O}(C)$ satisfying $|h(x_\rho)| < \rho^d$ for all $\rho \in]r_1,r_2[$ (where x_ρ denotes as usual the sup-norm on the annulus $\{|T| = \rho\}$). Since σ is an automorphism, we have $d = \pm 1$, and even $d = 1$, since it preserves the orientation.

As before, equality (2.88) is satisfied by the multiplications by q and q^{-1}, because they induce isomophisms of D_0 and D_∞. Replacing σ by $\sigma(q^{-1}\,\cdot)$, we can assume that $q = 1$.

In this case, the endomorphism σ stabilizes the sub-pseudo-annuli of C whose skeletons are included in Γ_C. Therefore, by Lemma 2.2.4, we can replace C by a smaller sub-pseudo-annulus containing b, and assume that $\mathscr{F}$ has log-affine radii along Γ_C.

By decomposing $\mathscr{F}$ by the radii and using Lemma 2.2.8, we can assume that the radii of $\mathscr{F}$ are all equal to the same log-affine affine function. We then distinguish two situations: $\mathscr{F} = \mathscr{F}^{<\mathrm{sol}}$ and

$\mathscr{F} = \mathscr{F}^{\mathrm{Robba}}$.

If $\mathscr{F} = \mathscr{F}^{<\mathrm{sol}}$, by Proposition 2.3.10, we have $\chi_b^{\mathrm{abs}}(\mathscr{F}) = \mathrm{Irr}_b(\mathscr{F})$, hence the claim follows from the fact that the radii are preserved by σ (see [PP15, Lemma 3.23]).

Assume now that $\mathscr{F} = \mathscr{F}^{\mathrm{Robba}}$. In this case Proposition A.1.5 shows that we have an isomorphism $\sigma_2^*(\mathscr{F}) \cong \mathscr{F}$. Equality (2.88) then follows from Lemma 2.2.7.

The last part of the statement follows from the first and from Remark 2.2.17, using the fact that two extensions of K may always be embedded in a common extension Ω that is spherically complete, algebraically closed and with $|\Omega| = \mathbb{R}_{\geqslant 0}$. □

With this result at hand, we can now extend our definition of absolute generalized index to a more general setting.

Definition 2.3.13. *Let X be a quasi-smooth K-analytic curve. Let b be a good germ of segment in X (cf. Definition 1.2.1) and let C be an open pseudo-annulus in X having b at its open boundary. Let $\mathscr{F}$ be a differential equation on X with log-affine radii along b. We say that $\mathscr{F}$ is Fredhlom at b if $\mathscr{F}_{|C}$ is and, in this case, we define the* absolute index of $\mathscr{F}$ at b *by*

$$\chi_b^{\mathrm{abs}}(\mathscr{F}) := \chi_b^{\mathrm{abs}}(\mathscr{F}_{|C}). \tag{2.91}$$

From what we have done, we know that the definition is independent of all the choices involved at the different steps of the construction: C (see Lemma 2.2.4), Ω (see Remark 2.2.17), the coordinates T_i (see Proposition 2.3.12) and the isomorphism j (see Lemma 2.2.7).

2.3.3. Log-affine radii

We now are ready to prove the index theorem over an annulus.

Theorem 2.3.14 (Index theorem). *Let C be an open pseudo-annulus and let b_0 and b_1 be the germs of segment at its open boundary. Let $\mathscr{F}$ be a differential equation over C whose radii are all log-affine along b_0 and b_1.*

For $b \in \{b_0, b_1\}$, let C_b be an open sub-pseudo-annulus of C containing b such that $\Gamma_{C_b} \subseteq \Gamma_C$ and such that $\mathscr{F}$ has log-affine radii along Γ_{C_b}.

Then, the following assertions are equivalent:

i) for each $b \in \{b_0, b_1\}$, $\mathscr{F}^{\mathrm{Robba}}_{|C_b}$ is Fredholm at b and we have

$$\chi_{b_0}^{\mathrm{abs}}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_0}}) \;=\; -\chi_{b_1}^{\mathrm{abs}}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_1}})\;; \tag{2.92}$$

ii) $\mathscr{F}$ has finite-dimensional de Rham cohomology over C and we have

$$\chi_{\mathrm{dR}}(C, \mathscr{F}) \;=\; \mathrm{Irr}_{b_0}(\mathscr{F}) + \mathrm{Irr}_{b_1}(\mathscr{F})\;. \tag{2.93}$$

Proof. By Lemma 2.2.18 and Proposition 2.3.10, i) implies ii) and, if K is not trivially valued, then ii) implies i).

In the case where K is trivially valued, it follows from Corollary A.3.2 that i) is always satisfied, hence ii) is alway satisfied too.

□

For further reference, let us extract from the preceding proof the statement in the trivially valued case. Note also that, by Remark 1.2.4, the assumption that the radii are log-affine is always satisfied in this case.

Theorem 2.3.15. *Assume that K is trivially valued. Let C be an open pseudo-annulus and let b_0 and b_1 be the germs of segment at its open boundary. Let $\mathscr{F}$ be a differential equation over C.*

Then, the radii of $\mathscr{F}$ are log-affine along b_0 and b_1, $\mathscr{F}$ has finite-dimensional de Rham cohomology over C and we have

$$\chi_{\mathrm{dR}}(C,\mathscr{F}) \;=\; \mathrm{Irr}_{b_0}(\mathscr{F})+\mathrm{Irr}_{b_1}(\mathscr{F})\;. \tag{2.94}$$

□

2.3.4. Non-log-affine radii

In order to generalize Theorem 2.3.14 to the case where the radii are not necessarily log-affine at the boundary, we introduce conditions on germs of segments.

Definition 2.3.16 (Condition $(\mathrm{Fin})_b^+$)**.** *Let X be a quasi-smooth K-analytic curve and let $\mathscr{F}$ be a differential equation on X. Let b be a good germ of segment in X. We say that $\mathscr{F}$* satisfies $(\mathrm{Fin})_b^+$ *if there exists an open pseudo-annulus C_b whose skeleton represents b such that*

- *i) $\mathscr{F}$ has log-affine radii along C_b;*
- *ii) $\mathscr{F}^{\mathrm{Robba}}_{|C_b}$ is Fredholm at b;*
- *iii) $\chi_b^{\mathrm{abs}}(\mathscr{F}^{\mathrm{Robba}}_{|C_b})=0$.*

Definition 2.3.17 (Condition $(\mathrm{Fin})_b$)**.** *Let X be a quasi-smooth K-analytic curve and let $\mathscr{F}$ be a differential equation on X. Let b be a good germ of segment in X. We say that $\mathscr{F}$* satisfies $(\mathrm{Fin})_b$ *if, for each open pseudo-annulus C whose skeleton represents b, there exists an open sub-pseudo-annulus $C'\subseteq C$ with $\Gamma_{C'}\subseteq\Gamma_C$ (possibly not representing b) such that $\mathscr{F}$ satisfies $(\mathrm{Fin})_{b'}^+$ where b' denotes the germ of segment at the open boundary of C' with the same orientation as b.*

Remark 2.3.18. *If K is trivially valued or, more generally, if K has residue characteristic 0, then, by Corollary A.3.2, each good germ of segment in X satisfies $(\mathrm{Fin})_b$.*

Corollary 2.3.19. *Let C be an open pseudo-annulus and let b_0 and b_1 be the germs of segment at its open boundary. Let $\mathscr{F}$ be a differential equation over C. Assume that $\mathscr{F}$ satisfies $(\mathrm{Fin})_{b_0}$ and $(\mathrm{Fin})_{b_1}$.*

Consider the following assertions are equivalent:

- *i) the total height of $\mathscr{F}$ is log-affine along b_0 and b_1;*
- *ii) for each $i\geqslant 0$, $\mathrm{H}^i_{\mathrm{dR}}(C,\mathscr{F})$ is finite dimensional.*

Moreover, when they hold, we have

$$\chi_{\mathrm{dR}}(C,\mathscr{F}) \;=\; \mathrm{Irr}_{b_0}(\mathscr{F})+\mathrm{Irr}_{b_1}(\mathscr{F})\;. \tag{2.95}$$

Proof. Let us first assume that K is not trivially valued. By assumption, there exists an increasing sequence $(C_n)_{n\geqslant 0}$ of relatively compact open pseudo-annuli of C with $\Gamma_{C_n}\subseteq\Gamma_C$ such that

$$\bigcup_{n\geqslant 0} C_n = C \tag{2.96}$$

and, for each $n\geqslant 0$, $\mathscr{F}$ satisfies $(\mathrm{Fin})_{b_{n,0}}$ and $(\mathrm{Fin})_{b_{n,1}}$, where $b_{n,0}$ (resp. $b_{n,1}$) denotes the germ of segment at the open boundary of C_n with the same orientation as b_0 (resp. b_1). By Theorem 2.3.14, for each $n\geqslant 0$, $\mathscr{F}_{|C_n}$ has finite dimensional de Rham cohomology and we have

$$\chi_{\mathrm{dR}}(C_n,\mathscr{F}_{|C_n})=\mathrm{Irr}_{b_{n,0}}(\mathscr{F})+\mathrm{Irr}_{b_{n,1}}(\mathscr{F}). \tag{2.97}$$

By log-concavity of the radii (see) , the sequences $(\mathrm{Irr}_{b_{n,0}}(\mathscr{F}))_{n\geqslant 0}$ and $(\mathrm{Irr}_{b_{n,1}}(\mathscr{F}))_{n\geqslant 0}$ are both non-increasing. In particular, the sequence $(\chi_{\mathrm{dR}}(C_n,\mathscr{F}_{|C_n}))_{n\geqslant 0}$ converges if, and only if, both sequences $(\mathrm{Irr}_{b_{n,0}}(\mathscr{F}))_{n\geqslant 0}$ and $(\mathrm{Irr}_{b_{n,1}}(\mathscr{F}))_{n\geqslant 0}$ converge, which is equivalent to saying that $\mathscr{F}$ has log-affine total height along b_0 and b_1 or, in other words, that $\mathscr{F}$ has well-defined irregularity. The result now follows from Theorem [?, CM].REF

Let us now assume that K is trivially valued. By Remark 1.2.4, the radii of $\mathscr{F}$ are log-affine along b_0 and b_1, hence assertion i) holds.

Let L be a complete valued extension of K with non-trivial valuation. The space C_L is a finite disjoint union of open pseudo-annuli $C_1, \dots, C_m$. For each $j \in \{1, \dots, m\}$, denote by $b_{j,0}$ (resp. $b_{j,1}$) the germ of segment of C_i over b_0 (resp. b_1). For each $j \in \{1, \dots, m\}$, the radii of $\mathscr{F}_L$ are log-affine along $b_{j,0}$ and $b_{j,1}$, hence it follows from the result in the non-trivially valued case that, for each $i \geqslant 0$, $\mathrm{H}^i_{\mathrm{dR}}(C_j, \mathscr{F}_L)$ is finite dimensional and that we have

$$\chi_{\mathrm{dR}}(C_j, \mathscr{F}_L) = \mathrm{Irr}_{b_{j,0}}(\mathscr{F}_L) + \mathrm{Irr}_{b_{j,1}}(\mathscr{F}_L). \tag{2.98}$$

It now follows from Theorem 1.4.7 that assertion ii) holds, as well as (A.29). □

2.4. Index formula for Robba rings

Let X be a quasi-smooth K-analytic curve. Let b be a good germ of segment in X.

Definition 2.4.1. *We call* Robba ring at b *the ring*

$$\mathfrak{R}_b \;:=\; \varinjlim_{C} \mathscr{O}(C)\;, \tag{2.99}$$

where C runs through the family of open pseudo-annuli whose open boundary contains b.

Let $\mathscr{F}$ be a differential equation over X. The restriction $\mathscr{F}_{|\mathfrak{R}_b}$ of $\mathscr{F}$ to $\mathfrak{R}_b$ is a locally free $\mathfrak{R}_b$-module of finite rank endowed with a connection.

Lemma 2.4.2. *For $i = 0, 1$, we have a natural isomorphism of K-vector spaces*

$$\varinjlim_{C} \mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}_{|C}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{R}_b, \mathscr{F}_{|\mathfrak{R}_b})\;, \tag{2.100}$$

where C runs through the set of pseudo-annuli whose open boundary contains b.

Proof. The result for $\mathrm{H}^1_{\mathrm{dR}}$ follows from the fact that cokernels and colimits commute. Since the colimit in (2.99) is filtered and kernels commute with filtered colimits, the result also holds for $\mathrm{H}^0_{\mathrm{dR}}$. □

Corollary 2.4.3. *Let $C_1 \supseteq C_2 \supseteq \cdots$ be a decreasing sequence of open pseudo-annuli all of them having b in their open boundary and such that $\bigcap_n C_n = \emptyset$. Consider the following conditions.*

i) $\mathscr{F}$ has log-affine radii along Γ_{C_1}, for each $n \geqslant 1$, $\mathscr{F}_{|C_n}$ has finite-dimensional de Rham cohomology and

$$\chi_{\mathrm{dR}}(C_n, \mathscr{F}_{|C_n}) = 0\;. \tag{2.101}$$

ii) There exists a complete valued extension L of K with non-trivial valuation such that, for each $n \geqslant 1$, $\mathscr{F}_{|C_n}$ and $(\mathscr{F}_L)_{|(C_n)_L}$ have finite-dimensional de Rham cohomologies and

$$\chi_{\mathrm{dR}}(C_n, \mathscr{F}_{|C_n}) \;=\; \chi_{\mathrm{dR}}(C_{n+1}, \mathscr{F}_{|C_{n+1}})\;. \tag{2.102}$$

Then i) implies ii). Moreover, if ii) holds, then for $i = 0, 1$, the cohomology group $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{R}_b, \mathscr{F}_{|\mathfrak{R}_b})$ is finite-dimensional and, for each $n \geqslant 1$, the natural maps

$$\mathrm{H}^i_{\mathrm{dR}}(C_n, \mathscr{F}_{|C_n}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(C_{n+1}, \mathscr{F}_{|C_{n+1}}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{R}_b, \mathscr{F}_{|\mathfrak{R}_b}) \tag{2.103}$$

are isomorphisms.

Proof. Assume that i) holds. If K is not trivially valued, then ii) holds with $L = K$. Assume that K is trivially valued. Let L be a complete non-trivially valued extension of K. Let $n \geqslant 1$. Then, by Corollary A.3.3, $(\mathscr{F}_L)_{|(C_n)_L}$ has finite-dimensional de Rham cohomology and we have $\chi_{\mathrm{dR}}((C_n)_L, \mathscr{F}_{|(C_n)_L}) = 0$. In particular, ii) holds.

Let us now assume that ii) holds. Let $n \geqslant 1$. For $i = 0, 1$, denote by $R_n^i : \mathrm{H}^i_{\mathrm{dR}}(C_n, \mathscr{F}_{|C_n}) \to \mathrm{H}^i_{\mathrm{dR}}(C_{n+1}, \mathscr{F}_{|C_{n+1}})$ the restriction map. For $i = 0$ it is injective, while for $i = 1$ it is surjective by Lemma 1.4.8. A dimension argument combined with (2.103) implies that R_n^0 and R_n^1 are isomorphisms. The claim now follows from Lemma 2.4.2. □

2.5. Formal differential equations.

In this section, we assume that the valuation of K is trivial and we describe more explicitly the consequences of Theorem 2.3.15.

Let $C := \{r_1 < |T| < r_2\}$, with $0 \leqslant r_1 < r_2 \leqslant +\infty$ be a standard open pseudo-annulus over K. Let $\mathscr{F}$ be a differential equation over C.

First of all, we recall that the triviality of the valuation implies that

$$\mathscr{O}(C) \;=\; \begin{cases} K((T)) & \text{if} \quad 0 \leqslant r_1 < r_2 \leqslant 1\,; \\ K[T, T^{-1}] & \text{if} \quad 0 \leqslant r_1 < 1 < r_2 \leqslant +\infty\,; \\ K((T^{-1})) & \text{if} \quad 1 \leqslant r_1 < r_2 \leqslant +\infty\,. \end{cases} \tag{2.104}$$

If $0 \leqslant r_1 < r_2 \leqslant 1$, we deduce that every analytic function over C naturally extends to the whole punctured disk $\{0 < |T| < 1\}$ and is bounded on all sub-annuli of the form $\{r < |T| < 1\}$, with $r > 0$. It follows that the differential equation $\mathscr{F}$ also extends to the whole punctured disk.

A similar phenomenon occurs in the other cases. Therefore, we can assume that $\mathscr{F}$ is defined either on $\{0 < |T| < 1\}$, $\{0 < |T| < +\infty\}$, or $\{1 < |T| < +\infty\}$.

In these three cases, the functions of $\mathscr{O}(C)$ are bounded in the neighborhood of the Gauss point $x_{0,1}$, hence it is a classical fact that it has a meaning to consider the restriction of functions of $\mathscr{O}(C)$ to the generic disk $D(x_{0,1})$, and therefore also the restriction of any differential equation $\mathscr{F}$ over C to $D(x_{0,1})$ [Chr83, ????].[5] In particular it has a meaning to consider the radii of $\mathscr{F}$ at $x_{0,1}$.

It follows easily from the triviality of the absolute value of K that we have $\mathcal{R}_i(x_{0,1}, \mathscr{F}) = 1$ for all i. This fact is known as *solvability* at $x_{0,1}$. In the case where $C = \{0 < |T| < \infty\}$ it implies that the radii can only have a break at $x_{0,1}$. More precisely, for every extension L/K and every point $x \in C(L)$, one verifies that the restriction of $\mathscr{F}$ to the maximal disk $D(x, S)$ is trivial. Hence the controlling graph of $\mathscr{F}$ is given by $\Gamma_\emptyset(\mathscr{F}) = \Gamma_C$ and the radii are all log-affine along the segments $]0, x_{0,1}]$ and $[x_{0,1}, \infty[$. Indeed the radii along Γ_C can be computed explicitly by the formal Newton polygon in a cyclic basis (see [PP13, Section 5.7], the proof is similar to that of Propositions 3.2.3 and 2.3.10).

Since the residue field of K has characteristic 0, by Corollary A.3.2, the equivalent conditions of Proposition 2.3.10 and Corollary 2.3.19 hold. More explicitly, in the case where $C = \{0 < |T| < 1\}$, we can identify $\mathscr{F}(C)$ to $K((T))^r$ and ∇ to a K-linear endomorphism of $K((T))$. Denote by b_0 and b_1 the germs of segment at the open boundary of C as in Section 2.2.1. Then with the notation

[5] For instance if t is an indeterminate over K we can consider the field extension $K \to K((t))$, and endow $K((t))$ with the absolute value $|\sum a_n t^n| = \sup |a_n|$. It follows that t is a generic point for $x_{0,1}$ (cf. [PP13, Notation 2.1.5]) , and any bounded function on C converges over the generic disk $D(x_{0,1})$. More concretely we have a ring homomorphism $\mathscr{O}(C) \to \mathscr{O}(\{|T - t| < 1\})$ defined by $f(T) \mapsto \sum_{n \geqslant 0} f^{(n)}(t)(T - t)^n / n!$, that commutes with d/dT.

In more geometric terms, since the absolute value associated to the Gauss point belongs to the spectrum of the ring of bounded functions, one may define a generic disk over this point by the usual base-change techniques (see [PP15, Section 2.2]).

(2.22) we have $V_0 := K[[T]]^r$ and $V_1 := T^{-1}K[T^{-1}]^r$, and Proposition 2.3.10 gives

$$\mathrm{Irr}^F_{b_0}(\mathscr{F}) \;=\; \chi^{\mathrm{abs}}_{b_0}(\mathscr{F}) \;:=\; \chi(K[[T]]^r, p_0 \circ \nabla(T\frac{d}{dT}) \circ i_0)\;, \tag{2.105}$$

$$\mathrm{Irr}^F_{b_1}(\mathscr{F}) \;=\; \chi^{\mathrm{abs}}_{b_1}(\mathscr{F}) \;:=\; \chi(T^{-1}K[T^{-1}]^r, p_1 \circ \nabla(T\frac{d}{dT}) \circ i_1)\;, \tag{2.106}$$

where the superscript ^{F} stands for "*Formal*" and indicates that we are computing the slopes of the radii with respect to the trivial valuation on K. In particular, since the radii are log-affine along $]0, x_{0,1}]$, one has

$$\mathrm{Irr}^F_{b_0}(\mathscr{F}) \;=\; -\,\mathrm{Irr}^F_{b_1}(\mathscr{F})\;. \tag{2.107}$$

In this situation, Theorem 2.3.15 gives another proof of the classical index theorem over $K((T))$ of Deligne-Malgrange [Mal74]: if $G(T) \in M_r(K((T)))$, then, for the connection $\nabla := d/dT - G(T)$ on $K((T))^r$, we have

$$\chi(K((T))^r, \nabla) \;=\; 0\;. \tag{2.108}$$

2.6. Some remarks about the boundary conditions

We now state some important remarks. In the following $\mathscr{F}$ is a differential equation on a quasi-smooth K-analytic curve X and C is an open pseudo-annulus in X such that the radii of $\mathscr{F}$ are all log-affine along Γ_C.

In this section, we deal with the conditions (Fin), the Liouville conditions on the exponents, and the absolute indexes. By definition, to study them, it is harmless to extend the base-field. As a consequence, we will assume that K is spherically complete, algebraically closed and that $|K| = \mathbb{R}_{\geqslant 0}$.

Over such a field, it follows from [Liu87] (cf. also [PP13, Remark 1.1.7]) that any pseudo-annulus is isomorphic to a standard one

$$C \;\cong\; \{r_1 < |T| < r_2\}\;, \qquad r_1, r_2 \in [0, +\infty]\;. \tag{2.109}$$

In this section, we will assume that C has the latter form with the standard orientation.

i) It is known that condition $(\mathrm{Fin})_C$ is automatically fulfilled in the following situations :
   (a) the characteristic of the residual field $\widetilde{K}$ of K is 0;
   (b) the characteristic of $\widetilde{K}$ is positive, the Christol-Mebkhout exponent of $\mathscr{F}^{\mathrm{Robba}}$ is non-Liouville and has non-Liouville differences (cf. Corollary A.3.2);
   (c) $\mathscr{F}^{\mathrm{Robba}}$ is an extension of rank one differential modules defined by equations $\frac{d}{dT} - g(T)$ whose residue $\mathrm{res}(g) \in \mathbb{Z}_p$ is non-Liouville.[6]

   Indeed, it is known that if (a) or (b) hold, then $\mathscr{F}$ satisfies (c) (cf. Theorems A.1.28 and A.3.1). Condition (c) implies condition $(\mathrm{Fin})_C$ because the absolute indexes are additive on exact sequences, therefore we are reduced to working with rank one differential equations, in which case the computation of the absolute index can be achieved directly (see for instance [CR94, Théorème 11.3.2] or [Rob75, Section 4.19]).

ii) If the characteristic of $\widetilde{K}$ is positive and if $\mathscr{F}^{\mathrm{Robba}}$ is extension of rank one modules $\{\mathscr{G}_i : \frac{d}{dT} - g_i(T)\}_{i=1,\dots,r}$, the exact condition for the finite dimensionality of the cohomology is that, for each i, the exponent $\mathrm{res}(g_i) \in \mathbb{Z}_p$ is non-Liouville. In particular, condition (b) is stronger than condition (c). The fact that the exponent of $\mathscr{F}^{\mathrm{Robba}}$ has non Liouville differences is

[6] Notice that a rank one equation of type Robba over C is always isomorphic to a differential equation $\mathscr{N}(\lambda)$ defined by an equation of the form $T\frac{d}{dT}(Y) = \lambda \cdot Y$, with $\lambda \in \mathbb{Z}_p$ (cf. for instance [Pul07, Lemma 1.3, Propositions 1.1, 1.2] and Corollary A.3.2).

actually used by Christol-Mebkhout to decompose $\mathscr{F}^{\mathrm{Robba}}$ into rank one pieces, but it is not a necessary condition for the finite dimensionality of the cohomology of $\mathscr{F}$.
However, if $\mathscr{F}^{\mathrm{Robba}} = 0$, and if the differential module $\mathrm{End}(\mathscr{F})$ has log-affine radii along Γ_C, then $\mathrm{End}(\mathscr{F})$ can have a non trivial Robba part whose nature seems relatively unrelated to the fact that $\mathscr{F}$ itself decomposes into rank one pieces. In this case we will see in section **??** that the condition

(b') the exponent of $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$ is non-Liouville and has non-Liouville differences

implies the essential algebraicity of $\mathscr{F}$ which is one of the key points of our index theorems in Section **??**. If $\mathscr{F}$ is not of type Robba (b') is essentially unrelated to condition (c).

iii) Denote by $\mathscr{N}(e)$ the differential module associated with the equation $T\frac{d}{dT}(Y) = e\cdot Y$, $e\in K$ (cf. Section A.1.3). Let b be a germ of segment in Γ_C. For $\lambda \in \mathbb{Z}_p$ we have $\mathrm{type}_b(\lambda) = 1$ (cf. Definition A.1.7) if, and only if, $\mathscr{N}(\lambda)$ is Fredholm at b, and in this case we have $\chi_b^{\mathrm{abs}}(\mathscr{N}(\lambda)) = 0$ (cf. Lemma A.1.12).
If b' is another germ of segment in Γ_C oriented as b, then, by definition, $\mathrm{type}_{b'}(\lambda) = \mathrm{type}_b(\lambda)$, therefore $\mathscr{N}(\lambda)$ is Fredholm at b if, and only if, it is Fredholm at b'. However, if b' has opposite orientation with respect to b, we may have $\mathrm{type}_{b'}(\lambda) < 1$.

iv) There are no examples where $\chi_b^{\mathrm{abs}}(\mathscr{F}^{\mathrm{Robba}})$ (resp. $\chi(C, \mathscr{F}^{\mathrm{Robba}})$) is non-zero but finite. To our knowledge, the only example of irreducible Robba module seems to be [Chr10] where a Robba module of rank two was considered.

v) If (a) (resp. (b), (c)) holds, then it still holds when $\mathscr{F}$ is replaced by some sub-quotient. In this case, $(\mathrm{Fin})_C$ and $(\mathrm{Fin})_b$ pass to sub-quotients as well.
In the general case it is unknown whether conditions $(\mathrm{Fin})_C$ and $(\mathrm{Fin})_b$ passes to sub-quotients.[7]
However, there are no examples where this fails.
We also recall that the condition $\chi_{\mathrm{dR}}(C, \mathscr{F}) = 0$ passes to the sub-quotients (cf. Lemma **??**).

vi) Condition $(\mathrm{Fin})_C$ is satisfied by $\mathscr{F}$ if the radii of $\mathscr{F}$ are all spectral non-solvable along Γ_C, because in this case $\mathscr{F}^{\mathrm{Robba}} = 0$.
Condition $(\mathrm{Fin})_C$ is also satisfied in the following situations:
  (A) Let D be a disk in X let b be the germ of segment at the open boundary of D. If the radii of $\mathscr{F}$ are all constant along b, then they are constant on the whole D by [Ked15, Lemma 4.3.12]. Therefore, $\mathscr{F}^{\mathrm{Robba}}$ is trivial and (c) is satisfied. It follows that $\mathscr{F}$ satisfies $(\mathrm{Fin})_b^+$.
  (B) Let $x\in X$. Condition $(\mathrm{Fin})_b^+$ holds at all germs of segment b out of x that are not in the controlling graph $\Gamma_S(\mathscr{F})$. Indeed, the connected component of $X - \Gamma_S(\mathscr{F})$ containing b is an open disk where (A) applies. In particular, $(\mathrm{Fin})_b$ may possibly fail only if $b \in \Gamma_S(\mathscr{F})$.
  (C) With the same notations as in (B), if all the radii are spectral non-solvable at x, then condition $(\mathrm{Fin})_b$ holds for all germs of segment b out of x. Indeed, if C_b is an open pseudo-annulus in X having b in its open boundary and such that the radii of $\mathscr{F}$ are all log-affine along Γ_{C_b}, then the Robba part of $\mathscr{F}_{|C_b}$ is trivial.

vii) The conditions of being Fredholm at b, the conditions (Fin), and the Liouville conditions on the exponents of $\mathscr{F}$ and $\mathrm{End}(\mathscr{F})$, are not stable by tensor product nor by internal Hom. For instance, with the notations of i)-(c) one has $\mathscr{N}(\lambda)\otimes\mathscr{N}(\lambda') = \mathscr{N}(\lambda+\lambda')$ and $\mathrm{Hom}(\mathscr{N}(\lambda), \mathscr{N}(\lambda')) = \mathscr{N}(\lambda' - \lambda)$, but, if λ and λ' are non-Liouville, then $\lambda+\lambda'$ and $\lambda'-\lambda$ are not necessarily non-Liouville.

[7] To prove this fact a natural strategy would be the following. Using Lemma 2.2.8, we may argue in a similar way as the proof of Lemma **??**. But this fails because the proof of Lemma **??** uses in an essential way the fact that the space of solutions of a differential equation (i.e. the kernel of the connection) is always finite dimensional, while in our context the kernels of the corresponding truncated operators (2.3) are not necessarily finite dimensional. A further argument is needed.

viii) It follows from Lemma 2.2.8 that if $0 \to \mathrm{M} \to \mathrm{E} \to \mathrm{N} \to 0$ is an exact sequence where M and N are both Fredholm at b (resp. both satisfy $(\mathrm{Fin})_C$), then so does E. This is also true for $(\mathrm{Fin})_b$ if one can take the same annuli C' for M and N in Definition **??**.

Let us now turn back to the case of an arbitrary field K, but assume that it has residue characteristic 0. Note that this covers the case where K is trivially valued. Then i)-(a) has many important consequences such as the fact that a differential equation over an open pseudo-annulus whose radii are log-linear radii at the boundary always has finite-dimensional de Rham cohomology (see Theorem 2.3.14). Similarly, many statements that we had in the previous sections can be simplified because their assumptions are automatically satisfied. As an example, let us rewrite Lemma 2.3.6 in this setting.

Lemma 2.6.1. *Assume that K has residue characteristic 0 (which holds for instance if K is trivially valued). Let C be an open pseudo-annulus and let $\mathscr{F}$ be a differential equation on C with log-affine radii at the boundary. Let C' be an open sub-pseudo-annulus of C with $\Gamma_{C'} \subseteq \Gamma_C$ such that $\chi_{\mathrm{dR}}(C,\mathscr{F}) = \chi_{\mathrm{dR}}(C',\mathscr{F}_{|C'})$. Then, for each $i \in \{0,1\}$, the restriction map*

$$\mathrm{H}^i_{\mathrm{dR}}(C,\mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(C',\mathscr{F}_{|C'}) \tag{2.110}$$

is an isomorphism.

3. Differential equations over an open disk with a meromorphic singularity

In this section, we provide an index formula for differential equations over open *disks* with some meromorphic singularities.

Such differential equations arise naturally in several contexts, but the study of their meromorphic cohomology from a global point of view had not been carried out so far. Some important classical developments are due to Clark [Cla66] and Baldassarri [Bal82] and concern differential equations over a *germ of punctured disk*, i.e. differential equations over the field of convergent Laurent power series $K(\{T\})$ (cf. Definition 3.5). In this context, their results are stated under the crucial assumption that the formal exponents at 0 are non-Liouville and/or have non-Liouville differences (the link with their results is given in Appendix 4).

The novelty of this section is precisely the fact that the exact necessary and sufficient condition for the finite dimensionality of the meromorphic de Rham cohomology over the disk is not a Liouville condition and does not arise at 0 (contrary to what the results of Clark might suggest). The exact condition arises at the open boundary of the disk, and it consists precisely in the fact that the equation is Fredholm at the open boundary of the disk (not at the meromorphic singularities).

3.1. Setting.

For the whole Section 3, we fix a positive real number $0 < r_D \leqslant +\infty$ and set

$$D := \{|T| < r_D\}\,; \tag{3.1}$$

$$C := \{0 < |T| < r_D\} \tag{3.2}$$

We set

$$b_D := \text{the germ of segment at the open boundary of } D\,, \tag{3.3}$$

$$b_0 := \text{the germ of segment out of } 0\,; \tag{3.4}$$

Denote by $K(\{T\})$ the field of convergent Laurent series. Every element of $K(\{T\})$ can be written as $\sum_{n\geqslant n_0} a_n T^n$, with $n_0 \in \mathbb{Z}$, for every $n \geqslant n_0$, $a_n \in K$ and the power series $\sum_{n\geqslant 0} |a_n|\, T^n$ has a positive radius of convergence. In other words, if D' runs in the set of open disks centered at 0, we have

$$K(\{T\}) \;=\; \varinjlim_{D'} \mathscr{O}(D')[T^{-1}] \;=\; \bigcup_{D'} \mathscr{O}(D')[T^{-1}]\;. \tag{3.5}$$

We also consider the Robba ring at 0

$$\mathfrak{R}_0 \;:=\; \mathfrak{R}_{b_0} \;=\; \varinjlim_{C'} \mathscr{O}(C') \;=\; \bigcup_{C'} \mathscr{O}(C')\;, \tag{3.6}$$

where C' runs in the set of pseudo-annuli of the form $\{0 < |T| < r\}$ with $r > 0$.

The intersection of $K(\{T\})$ with $\mathscr{O}(C)$ in $\mathfrak{R}_0$ is the ring $\mathscr{O}(D)[T^{-1}]$ obtained from the ring $\mathscr{O}(D)$ of analytic functions on D by inverting T.

We have a diagram of inclusions of rings

$$\begin{array}{ccccc} \mathscr{O}(D)[T^{-1}] & \subset & K(\{T\}) & \subset & K((T)) \\ \cap & & \cap & & \\ \mathscr{O}(C) & \subset & \mathfrak{R}_0 & & \end{array} \tag{3.7}$$

where all the inclusions commute with d/dT.

Remark 3.1.1. *Notice that if the valuation of K is trivial we have $\mathscr{O}(D)[T^{-1}] = \mathscr{O}(C)$ and $\mathfrak{R}_0 = K(\{T\}) = K((T))$; if moreover $r_D \leqslant 1$ then all the above rings coincide.*

In accordance with Section 1.5, the notation $D(*0)$ indicates the disk D with structure sheaf $\mathscr{O}_D[*0]$ formed by analytic functions on $D - \{0\}$ with a meromorphic pole at 0. The ring of global sections of $\mathscr{O}_D[*0]$ is $\mathscr{O}(D)[T^{-1}]$.

According to Definition 1.5.1, a (meromorphic) differential equation $\mathcal{F}$ over $D(*0)$ is a sheaf of locally free $\mathscr{O}_D[*0]$-modules of finite rank on D endowed with a connection. By abuse of notation we often denote by $\mathcal{F}$ the $\mathscr{O}(D)[T^{-1}]$-module of its global sections. This does not cause any trouble because of the following result.

Proposition 3.1.2. *The global section functor sets up an equivalence between the category of locally free $\mathscr{O}_D[*0]$-modules of finite rank (resp. endowed with a meromorphic connection) and the category of projective $\mathscr{O}(D)[T^{-1}]$-modules of finite type (resp. endowed with a connection).* □

Proof. Such a result is classical if one replaces $\mathscr{O}_D[*0]$ and $\mathscr{O}(D)[T^{-1}]$ by $\mathscr{O}_D$ and $\mathscr{O}(D)$ respectively (see [**?**, Corollary 4.11] for instance). The proof relies on Kiehl's Theorem: for every coherent sheaf $\mathscr{F}$ of $\mathscr{O}_D$-modules, one has $H^1(D, \mathscr{F}) = 0$ and $\mathscr{F}$ is generated by its global sections. Note that Kiehl's Theorem immediately extends to $\mathscr{O}_D$-modules that are filtered direct limits of coherent sheaves. Arguing as in the classical case then gives the result we want. The version with connections follows immediately. We refer to [**?**, Corollary 1.7.15] for more details. □

For a differential equation $\mathcal{F}$ over $D(*0)$ we set

$$\mathscr{F} \;:=\; \mathcal{F} \otimes_{\mathscr{O}_D[*0]} \mathscr{O}_C \;=\; \mathcal{F}_{|C}\;, \tag{3.8}$$

$$\mathcal{F}_0^{\dagger} \;:=\; \mathcal{F} \otimes_{\mathscr{O}_D[*0]} K(\{T\})\;, \tag{3.9}$$

$$\mathrm{M} \;:=\; \mathcal{F} \otimes_{\mathscr{O}_D[*0]} K((T))\;, \tag{3.10}$$

$$\mathscr{F}_0^{\dagger} \;:=\; \mathcal{F} \otimes_{\mathscr{O}_D[*0]} \mathfrak{R}_0\;. \tag{3.11}$$

3.2. Log-affinity of the radii at 0.

We begin by showing that the radii of $\mathscr{F}$ are all log-affine around 0.

For $\rho \in]0, r_D[$, denote by x_ρ as usual the point unique point in the Shilov boundary of the annulus $\{|T| = \rho\}$.

If, in a cyclic basis, $\mathscr{F}$ is given by a differential operator $\mathscr{L}$, there is a (spectral Newton) polygon associated with $\mathscr{L}$ whose slopes are related to the radii of the solutions by Young's theorem (cf. [You92] and [Pul15, Proposition 4.11] for a setting more closely to ours). The notation of [Pul15, Proposition 4.11] is adapted to spectral radii [Pul15, (4.4)], therefore we here reformulate the statement with respect to the normalized radii $\mathcal{R}_i(x_\rho, \mathscr{F})$ (cf. [PP13, Definition 2.3.1]).

Definition 3.2.1. *Let $\mathscr{L} := \sum_{i=0}^{r} g_{r-i}(T) \cdot (d/dT)^i$ with $g_0 = 1$ and, for all $j \in \{1, \dots, r\}$, $g_j \in \mathscr{O}(C)$. Denote by $\mathscr{F}$ the differential equation associated with $\mathscr{L}$. Define the spectral Newton polygon $NP(x_\rho, \mathscr{L})$ of $\mathscr{L}$ at x_ρ as the Newton polygon of the set (cf. (1.10))*

$$\left\{ \left(k, -\ln\left((\rho/\omega)^k \cdot |g_k(x_\rho)|\right)\right) \mid 0 \leqslant k \leqslant r \right\} . \tag{3.12}$$

Denote by $s_i^{\mathscr{L}}(x_\rho)$ the i-th slope of this polygon (cf. Definition 1.3.1). Set, as usual (cf. (1.14))

$$s_i(x_\rho) \ := \ \ln(\mathcal{R}_i(x_\rho, \mathscr{F})) . \tag{3.13}$$

It follows from the definition that

$$s_1^{\mathscr{L}}(x_\rho) \ = \ \min_{1 \leqslant k \leqslant r} \frac{-\ln\left((\rho/\omega)^k |g_k(x_\rho)|\right)}{k} . \tag{3.14}$$

Recall the definition of ω (cf. (0.1)).

Proposition 3.2.2 ([You92], [Pul15, Proposition 4.11])**.** *We maintain the notations of Definition 3.2.1. One has $s_i(x_\rho) < \ln(\omega)$ if, and only if, $s_i^{\mathscr{L}}(x_\rho) < \ln(\omega)$ and, in this case,*

$$s_i^{\mathscr{L}}(x_\rho) \ = \ s_i(x_\rho) . \qquad \square \tag{3.15}$$

Lemma 3.2.3. *There exists $\varepsilon \in]0, 1[$ such that the radii $\mathcal{R}_i(-, \mathscr{F})$ are all* log-affine *on the segment $\{x_\rho \mid 0 < \rho < \varepsilon\}$.*

Proof. By assumption $\mathcal{F}$ is a locally free $\mathscr{O}_D[*0]$-module, hence, by shrinking C around 0, we may assume that $\mathcal{F}$ is free. By shrinking again, we may assume that it has a cyclic basis where it is given by a differential operator $\mathscr{L}$. This operation does not affect the radii of $\mathscr{F}$. The advantage is that we can now read the radii in terms of the coefficients of $\mathscr{L}$.

Set $r := \mathrm{rank}(\mathscr{F})$. For all $i \in \{1, \dots, r\}$, the function $\log(\rho) \mapsto \log(H_i(x_\rho, \mathscr{F}))$ is concave along $]-\infty, \log(r_D)[$ and bounded by 0. We deduce that either $H_i(-, \mathscr{F})$ is constant on $]0, x_{\varepsilon'}[$, for some $\varepsilon' \in]0, r_D]$, or we have $\lim_{\log(\rho) \to -\infty} \log(H_i(x_\rho, \mathscr{F})) = -\infty$.

We now proceed by induction on i to show that the sequence of slopes of $\log(\rho) \mapsto \log(H_i(x_\rho, \mathscr{F}))$ is constant around 0. For $i = 1$, if we have $\lim_{\log(\rho) \to -\infty} \log(H_1(x_\rho, \mathscr{F})) = -\infty$, then for ρ close enough to 0, the first radius $\mathcal{R}_1(x_\rho, \mathscr{F}) = H_1(x_\rho, \mathscr{F})$ is smaller than $\ln(\omega)$, hence explicitly intelligible in terms of the coefficients of the operator $\mathscr{L}$, by Proposition 3.2.2. Since the coefficients lie in $K(\{T\})$, they have only finitely many slopes along $]0, x_{r_D}]$ and the result follows.

Now, assume inductively that for all $j \in \{1, \dots, i-1\}$ the radii $\mathcal{R}_j(-, \mathscr{F})$ are log-affine on some $]0, x_{\varepsilon'}[$. Since $\log(H_i(-, \mathscr{F}))$ is concave, then so is

$$\log(\mathcal{R}_i(-, \mathscr{F})) = \log(H_i(-, \mathscr{F})) - \sum_{j=1}^{i-1} \log(\mathcal{R}_j(-, \mathscr{F})) \tag{3.16}$$

and we can use the same argument as above. □

Corollary 3.2.4. *Let P be a quasi-smooth K-analytic curve, let z be a rigid point in P and let $\mathcal{G}$ be a meromorphic differential equation on P with poles at z. Set $Y := P - \{z\}$ and denote by b_z the germ of segment out of the point z, seen as a germ of segment at the open boundary of Y. Then all the radii of convergence of $\mathscr{G} := \mathcal{G}_{|Y}$ are log-affine along b_z (in the sense of Definition 1.2.3).* □

Definition 3.2.5. *In the setting of Corollary 3.2.4, we define the irregularity of $\mathcal{G}$ at z to be*

$$\mathrm{Irr}_z(\mathcal{G}) \ := \ \mathrm{Irr}_{b_z}(\mathscr{G}) \ \in \ \mathbb{Z} \tag{3.17}$$

(see Definition 1.2.5).

Corollary 3.2.6. *Let $\mathfrak{Y}$ be a smooth connected algebraic curve over K and let $\mathfrak{F}$ be an algebraic differential equation on $\mathfrak{Y}$. Then, the analytified equation $\mathfrak{F}^{\mathrm{an}}$ has log-affine radii at the open boundary of $\mathfrak{Y}^{\mathrm{an}}$.* □

Proof. Consider a compactification $\mathfrak{Y}'$ of $\mathfrak{Y}$, *i.e.* a smooth connected projective curve over K containing $\mathfrak{Y}$ as an open subset. Then, $\mathfrak{F}$ extends to a differential equation $\mathfrak{F}'$ on $\mathfrak{Y}'$ with meromorphic poles on $\mathfrak{Z} := \mathfrak{Y}' - \mathfrak{Y}$. The germs of segment at infinity of $\mathfrak{Y}^{\mathrm{an}}$ correspond bijectively to the germs of segment in $\mathfrak{Y}'^{\mathrm{an}}$ out of the points of $\mathfrak{Z}^{\mathrm{an}}$. By Corollary 3.2.4, the differential equation $\mathfrak{F}^{\mathrm{an}}$ has log-affine radii along those germs. □

3.3. Derived Newton polygon at b_0.

In Section 1.3, we explained that there is no natural way to define a derivative of the convergence Newton polygon on a germ of segment. We here show that there is a natural definition at the germ of segment out of a point of type 1 or out of a meromorphic singularity.

Lemma 3.3.1. *Let $n \leqslant m \in \mathbb{Z}$. For each $i \in \{n, \dots, m\}$, let $c_i \in \mathbb{R} \cup \{+\infty\}$ and $p_i \in \mathbb{R}$. Assume that $c_n, c_m < +\infty$. Let $t \in \mathbb{R}$. For each $i \in \{n, \dots, m\}$, set $v_i := c_i + p_i t$ and denote by $N(t)$ the Newton polygon of the set $\{(i, v_i(t)) \mid n \leqslant i \leqslant m\}$. Denote its slopes by*

$$s_1(t) \ \leqslant \ \cdots \ \leqslant \ s_{m-n}(t)\ . \tag{3.18}$$

For each $i \in \{n, \dots, m\}$, set $\partial_{b_0} v_i := -\infty$ if $c_i = +\infty$ and

$$\partial_{b_0} v_i \ := \ \frac{d}{dt}(v_i(t)) \ = \ p_i \tag{3.19}$$

otherwise. Denote by

$$\partial_{b_0} N \ : \ [n, m] \ \longrightarrow \ \mathbb{R} \tag{3.20}$$

the inverted Newton polygon of the set $\{(i, \partial_{b_0} v_i) \mid n \leqslant i \leqslant m\}$. Denote its slopes by

$$s_1^{b_0} \ \geqslant \ \cdots \ \geqslant \ s_{m-n}^{b_0}\ . \tag{3.21}$$

Then, there exists $t_0 \in \mathbb{R}$, such that, for each $j \in \{1, \dots, m-n\}$, the map $t \in]-\infty, t_0[\mapsto s_j(t)$ is affine with slope $s_j^{b_0}$:

$$\frac{d}{dt}(s_j(t)) \ = \ s_j^{b_0}\ . \tag{3.22}$$

Proof. Let us prove the result by induction on $m - n$. If $m - n = 0$, there is no slope and the result is trivial.

Assume that $m > n$. Set $I := \{i \in \{n+1, \dots, m\} \mid c_i < +\infty\}$. Let $t \in \mathbb{R}$. The first slope of $N(t)$

is

$$s_1(t) = \min_{i\in I}\left(\frac{(c_i-c_n)+(p_i-p_n)t}{i-n}\right). \tag{3.23}$$

Set $P := \max_{i\in I}\left(\frac{p_i-p_n}{i-n}\right)$ and $I_p := \{i \in I \mid (p_i-p_n)/(i-n) = P\}$. Set $C := \min_{i\in I_p}\left(\frac{c_i-c_n}{i-n}\right)$. It is easy to check that there exists $t_0 \in \mathbb{R}$ such that, for each $t < t_0$, we have

$$s_1(t) \;=\; C + Pt\,. \tag{3.24}$$

Similarly, the first slope of $\partial_{b_0} N$ is

$$s_1^{b_0} = \max_{i\in I}\left(\frac{p_i-p_n}{i-n}\right) = P, \tag{3.25}$$

hence the result holds for the first slope.

Let $t < t_0$. The polygon $N(t)$ passes through the point $(n+1, c_n + p_n t + s_1(t))$. Set $\bar{v}_{n+1} := c_n+p_nt+s_1(t) = (c_n+C)+(p_n+P)t$. For $i \in \{n+2,\dots,b\}$, set $\bar{v}_i = v_i$. Denote by $\bar{N}(t)$ the Newton polygon of the set $\{(i,\bar{v}_i(t)) \mid n+1 \leqslant i \leqslant m\}$. By construction, its slopes $\bar{s}_1(t) \leqslant \cdots \leqslant \bar{s}_{m-n-1}(t)$ are exactly $s_2(t) \leqslant \cdots \leqslant s_{m-n}(t)$.

Similarly, the polygon $\partial_{b_0} N$ passes through the point $(n+1, p_n + s_1^{b_0})$. Set $\bar{v}_{n+1}^{b_0} := p_n + s_1^{b_0} = p_n + P$. For $i \in \{n+2,\dots,m\}$, set $\bar{v}_i^{b_0} = \partial_{b_0} v_i$. Denote by $\bar{N}^{b_0}$ the inverted Newton polygon of the set $\{(i,\bar{v}_i^{b_0}) \mid n+1 \leqslant i \leqslant m\}$. By construction, its slopes $\bar{s}_1^{b_0} \geqslant \cdots \geqslant \bar{s}_{m-n-1}^{b_0}$ are exactly $s_2^{b_0} \geqslant \cdots \geqslant s_{m-n}^{b_0}$.

Moreover, the polygon $\bar{N}^{b_0}$ is exactly the inverted Newton polygon associated with the Newton polygon $\bar{N}(t)$ by the construction of the statement. By induction, there exists $t_1 \leqslant t_0$ such that, for each $j \in \{1,\dots,m-n-1\}$, the map $t \in \,]-\infty, t_1[\, \mapsto \bar{s}_j(t)$ is affine with slope $\bar{s}_j^{b_0}$. This concludes the proof. $\square$

The following result is a direct consequence of Lemmas 3.2.3 and 3.3.1.

Proposition 3.3.2. *Let Z be a locally finite set of rigid points in X. Let $\mathcal{F}$ be a meromorphic differential equation on X with poles in $Z \subset X$. Let $x \in X$ be a point of type 1 and let b_x be the germ of segment out of x (as usual oriented away from x).*

Let C_x be an open pseudo-annulus whose skeleton represents b_x. In the following, we compute the radii of convergence on C_x. Let r be the rank of $\mathscr{F}$ on C_x.

Denote by

$$s_1^{b_x} \;\geqslant\; \cdots \;\geqslant\; s_r^{b_x} \tag{3.26}$$

the slopes of the inverted *polygon associated to the set*

$$\{(i, \partial_{b_x} NP(-,\mathscr{F})(i))\;,\; 0 \leqslant i \leqslant r\}\;, \tag{3.27}$$

where $NP(y,\mathscr{F})$ denotes the convergence Newton polygon of $\mathscr{F}$ at y (see Definition 1.3.3).

Then, for each $i \in \{1,\dots,r\}$, we have

$$\partial_{b_x} \ln(\mathcal{R}_i(-,\mathscr{F})) \;=\; s_i^{b_x}\,. \tag{3.28}$$

In particular, the sequence $(\partial_{b_x}(\ln(\mathcal{R}_1(-,\mathscr{F}))),\dots,\partial_{b_x}(\ln(\mathcal{R}_r(-,\mathscr{F}))))$ is non-increasing and non-negative.

$\square$

Proposition 3.3.2 shows that the two ways to construct a derived polygon from the convergence Newton polygon that we have evoked in Section 1.3 are actually the same. We are then allowed to formulate without ambiguity the following definition.

Definition 3.3.3 (Derivative of $NP(-,\mathscr{F})$)**.** *Let Z be a locally finite set of rigid points in X. Let $\mathcal{F}$ be a meromorphic differential equation on X with poles in $Z \subset X$. Let $x \in X$ be a point of type* 1 *and let b_x be the germ of segment out of x (as usual oriented away from x). We call* derivative of the convergence Newton polygon of $\mathcal{F}$ at b_x *the inverted polygon associated with the family* (1.16)*.*

Remark 3.3.4. *We maintain the notation of Definition 3.3.3. Assume that the derivative of the convergence Newton polygon of $\mathcal{F}$ at b_x has a break at $k \in \{1,\ldots,r-1\}$. Note that the point x must then belong to Z because the radii are constant around a type 1 point that is not a singularity of the differential equation.*

By Proposition 3.3.2, we $\partial_{b_x}(s_k) > \partial_{b_x}(s_{k+1})$. We also have $s_k < s_{k+1}$ over b_x, because the functions $s_k = \ln(\mathcal{R}_{S,k}(-,\mathscr{F}))$ are log-affine functions in the neighborhood of x (which corresponds to $-\infty$ in the logarithmic coordinate). Note that, for each y in some segment representing b_x, the convergence Newton polygon $NP_S(y,\mathscr{F})$ has a break at k too.

3.4. Analytic vs. formal irregularities.

We now compare formal and analytic irregularities at 0.

We maintain the notations of Section 3.1. In particular $\mathrm{M} = \mathcal{F} \otimes K((T))$.

We have seen in Section 2.5 that, if K is endowed with the trivial valuation, then $K((T))$ coincides with the ring $\mathscr{O}(C')$ of analytic functions over an annulus $C' = \{0 < |T| < r'\}$, with $r' \in]0,1[$. By Theorem 2.3.15, the index of M on $K((T))$ exists and it is always zero. On the other hand, Proposition 2.3.10 computes the absolute index of M at a germ of segment b in the skeleton of the formal annulus $\{0 < |T| < 1\}$ (cf. Section 2.5):

$$\chi_b^{\mathrm{abs}}(\mathrm{M}) \;=\; \mathrm{Irr}_b^F(\mathrm{M})\;, \tag{3.29}$$

where the superscript F (which stands for “Formal”) indicates that we are computing the slope with respect to the trivial valuation on K.

It is natural to ask whether the the irregularity $\mathrm{Irr}_0(\mathscr{F})$ from Definition 3.2.5 coincides with the formal irregularity $\mathrm{Irr}_0^F(\mathrm{M})$ (from Definition 3.2.5 again but applied to K endowed with the trivial valuation). We prove that this is indeed the case.

More specifically, if K is trivially valued, we denote by b_0^F the germ of segment out of 0 (as usual the superscript ^{F} stands for “formal”).

Propositions 3.3.2 and 3.2.2 applies to both $\mathcal{F}$ and M. The latter is a $K((T))$-differential module (cf. Remark 3.1.1). Now, there is another polygon classically associated with M: the *formal Newton polygon* that is defined in [PP13, Section 5.7]. The following result provides a link between all the polygons.

Proposition 3.4.1. *We maintain the notations of Section 3.1. The following polygons coincide:*

i) the derivative of the convergence Newton polygon of $\mathcal{F}$ at b_0 (cf. Definition 3.3.3);

ii) the derivative of the convergence Newton polygon of M *at b_0^F (cf. Definition 3.3.3 with K trivially valued).*

Moreover

iii) if $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_r$ are the slopes of the above polygons, then the formal Newton polygon of M *coincides with the polygon $N : [0,r] \to \mathbb{R}$ having slopes $p'_1 \leqslant p'_2 \leqslant \cdots \leqslant p'_r$, where $p'_i := p_{r-i+1}$ (i.e. reordered in increasing order) and such that $v_r = 0$.*

We summarize these properties by saying that the derivative of the convergence Newton polygon along b_0 is independent of the valuation of K. In particular,

iv) we have the equality

$$\mathrm{Irr}_{b_0}(\mathscr{F}) \;=\; \mathrm{Irr}_{b_0^F}(\mathrm{M}) \;. \tag{3.30}$$

Moreover, $\mathrm{Irr}_{b_0^F}(\mathrm{M})$ *coincides with the opposite of the formal irregularity* $i_0(\mathrm{M})$ *of* M *(cf. (??)) defined in [Ram78] and [DMR07] (cf. [PP13, Section 5.7]).*

v) If the formal Newton polygon of M *has a break at k, then so has the convergence Newton polygon $NP(y, \mathscr{F})$, for all y in some segment representing b_0.*

Proof. The claim is invariant by restriction of the radius of D. Therefore, we may assume that $\mathcal{F}$ is associated with a differential operator $\mathscr{L} := \sum_{i=0}^{r} g_{r-i}(T)(d/dT)^i$ with coefficients in $\mathscr{O}(*D)$ as in Proposition 3.2.2. Denote by $s_i^{\mathscr{L}}(x_\rho)$ (resp. $s_i^{L}(x_\rho)$) the i-th slope of this polygon at x_ρ. By Proposition 3.2.2, $s_i(x_\rho) := \ln(\mathcal{R}_i(x_\rho, \mathscr{F})) < \ln(\omega)$ if and only if $s_i^{\mathscr{L}}(x_\rho) < \ln(\omega)$ and in this case

$$s_i^{\mathscr{L}}(x_\rho) \;=\; \ln(\mathcal{R}_i(x_\rho, \mathscr{F})) \;=\; s_i(x_\rho) \;. \tag{3.31}$$

The same holds for L and M. Namely, if we set $s_i^F(x_\rho) := \ln(\mathcal{R}_i(x_\rho, \mathrm{M}))$, then $s_i^F(x_\rho) < \ln(\omega) = 0$ if and only if $s_i^L(x_\rho) < \ln(\omega) = 0$ and in this case one has $s_i^{\mathscr{L}}(x_\rho) \;=\; \ln(\mathcal{R}_i(x_\rho, \mathrm{M})) \;=\; s_i^F(x_\rho)$.

This correspondence of slopes holds only for slopes that are less than $\ln(\omega)$. In our case this is enough because the slopes s_i (resp. s_i^F) that are larger than $\ln(\omega)$ along b_0 (resp. b_0^F) are constant functions in a neighborhood of $-\infty$. Indeed, by Lemma 3.2.3 the functions s_i and s_i^F are affine along in a neighborhood of $-\infty$ and, by definition of the radii, they are also less than or equal to 0. It follows that if s_i (resp. s_i^F) is not constant over b_0 (resp. b_0^F), then it has to tend to $-\infty$ as ρ approaches 0. In particular, it is less than $\ln(\omega)$ in a neighborhood of $-\infty$ (see the proof of Lemma 3.2.3 for more details).

Now, denote by $v_i^{\mathscr{L}}$ (resp. v_i^L) the i-th partial height of $NP(x_\rho, \mathscr{L})$ (resp. $NP(x_\rho, L)$) as in Definition 3.2.1. For $i = 0, \ldots, r$, we denote by $s_i^{\mathscr{L}, b_0}$ (resp. s_i^{L, b_0^F}) the i-th slope of the Newton polygon associated with the set $\{(i, \partial_{b_0} v_i^{\mathscr{L}}\;,\; 0 \leqslant i \leqslant r)\}$ (resp. $\{(i, \partial_{b_0} v_i^L)\;,\; 0 \leqslant i \leqslant r\}$).

With this notation, to prove the equality of polygons as in i) and ii) it is enough to show that for all i one has $\partial_{b_0} s_i^{\mathscr{L}} = \partial_{b_0} s_i^L$. By Proposition 3.3.1, this is equivalent to showing that for all i one has

$$s_i^{\mathscr{L}, b_0} \;=\; s_i^{L, b_0^F} \;. \tag{3.32}$$

Now, this follows from the following remark. For each non-zero function $f(T) = \sum_i a_i T^i \in \mathscr{O}(D)[T^{-1}]$ and each $\rho > 0$ close enough to 0, one has

$$|f|(x_\rho) \;=\; |a_{v_T(f)}|\, \rho^{v_T(f)} \;, \tag{3.33}$$

where $v_T(f) = \min(i, a_i \neq 0)$ is the T-adic valuation of f. This equality is true for any valuation of K. In particular, the derivative $\partial_{b_0}(f)$ is independent of the valuation of K.

It follows that for all $i = 1, \ldots, r$ one has $\partial_{b_0}(g_i) = \partial_{b_0^F}(g_i)$ and therefore (3.32) holds. This proves the coincidence of the two polygons in i) and ii).

Item iii) follows from [PP13, Section 5.7]. Now, equality (3.30) follows readily from the coincidence of the polygons in i) and ii) because the irregularity is nothing but the opposite of their total height. Analogously, the equality between $\mathrm{Irr}_{b_0^F}(\mathrm{M})$ and $-i_0(\mathrm{M})$ follows from iii). Indeed, since the total heights of the polygons in ii) and of the formal Newton polygon in iii) coincide, the claim follows from Remark 1.3.4 and the fact that the total height of the formal Newton polygon is by definition $i_0(\mathrm{M})$.

Finally, v) follows from Remark 3.3.4. □

Remark 3.4.2. *In general, there is no direct relationship between $NP(-,\mathscr{F})$ and $NP(-,\mathrm{M})$. For instance, let $a\in K$ and let $\mathcal{F}$ be the equation $T\frac{d}{dT}(y)=ay$. If the valuation of K is trivial, then the formal radii of M are uniformly equal to 1 along the (formal) segment $]0,+\infty[$. Now, if the valuation of K is not trivial on $\mathbb{Z}$, and if the residue characteristic of K is p, then the radii of $\mathscr{F}$ along the segment $]0,+\infty[$ are constant and equal to $|p|^{\frac{1}{p-1}}\cdot\liminf_s|a(a-1)(a-2)\cdots(a-s+1)|^{-1/s}$ (cf. [Pul15, Lemma 1.4]). This radius is not equal to 1, i.e. non-maximal, if and only if $a\notin\mathbb{Z}_p$, (cf. [DGS94, Proposition 7.3, Chapter IV]).*

3.5. Index of a differential equation with meromorphic singularities on an open pseudo-disk.

In this section, we temporarily substitute the notation of Section 3.1 and we use the following setting instead.

Setting 3.5.1. *Let D be an open pseudo-disk (cf. [PP13, Definition 1.1.8]) and assume that there exists a finite extension L of K such that $D\otimes_K L$ is a (finite) disjoint union of open disks and affine lines over L.*[8] *Denote by b_D the germ of segment at the open boundary of D. Let Z be a finite set of rigid points of D and let $\mathcal{F}$ be a differential equation on $D(*Z)$.*

We set $Y:=D-Z$, $\mathscr{F}:=\mathcal{F}_{|Y}$ and

$$\mathrm{Irr}_Y(\mathscr{F}) \;=\; -\sum_{z\in Z}\mathrm{Irr}_z(\mathscr{F})-\mathrm{Irr}_{b_D}(\mathscr{F})\;. \tag{3.34}$$

Theorem 3.5.2. *We set notations as in Setting 3.5.1 and we assume moreover that*

i) $\mathcal{F}$ is a free $\mathscr{O}(D)[*Z]$-module;

ii) the radii of $\mathscr{F}$ are all log-affine along the germ of segment b_D at the open boundary of D.

Let C_{b_D} be an open pseudo-annulus in Y containing b_D such that $\mathscr{F}$ has log-affine radii along $\Gamma_{C_{b_D}}$.

Then, the following conditions are equivalent:

(a) $\mathcal{F}$ has finite-dimensional meromorphic cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(D(*Z),\mathcal{F})$;

(b) $\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}$ is Fredholm at b_D.

Moreover, in this case we have

$$\chi_{\mathrm{dR}}(D(*Z),\mathcal{F}) \;=\; \chi_c(Y)\cdot\mathrm{rank}(\mathcal{F})-\mathrm{Irr}_Y(\mathscr{F})+\chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}})\;. \tag{3.35}$$

In particular, the following assertions are equivalent:

(c) one has

$$\chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}) \;=\; 0\;; \tag{3.36}$$

(d) the index formula holds:

$$\chi_{\mathrm{dR}}(D(*Z),\mathcal{F}) \;=\; \chi_c(Y)\cdot\mathrm{rank}(\mathcal{F})-\mathrm{Irr}_Y(\mathscr{F})\;. \tag{3.37}$$

Proof. The open pseudo-disk D is cohomologically Stein, which means that coherent sheaves have no higher cohomology on it. This holds in particular for Ω^1_D. Since $\mathcal{F}$ may be written as a direct limit of coherent sheaves, the same result holds for it. A spectral sequence argument now shows

[8] Recall that an open pseudo-disk is a connected K-analytic curve that is not compact and has empty analytic skeleton. In particular D has genus 0 and $\chi_c(D)=1$. By [PP13, Lemma 1.1.26], for every non-trivially valued algebraically closed maximally complete extension M of K, $D\otimes_K M$ is a disjoint union of open disks and affine lines but it could happen that this decomposition does not descend to any finite extension of K.

that $\mathrm{H}^0_{\mathrm{dR}}(D(*Z),\mathcal{F})$ and $\mathrm{H}^1_{\mathrm{dR}}(D(*Z),\mathcal{F})$ coincide respectively with the kernel and the cokernel of the morphism

$$\nabla : \mathcal{F}(D) \;\to\; \mathcal{F}(D) \otimes_{\mathscr{O}(D)} \Omega^1_D(D)\ . \tag{3.38}$$

Let K' be a finite extension of K. Set $D' := D \otimes_K K'$ and denote by $\mathcal{F}'$ the pull-back of $\mathcal{F}$ on D' and by Z' the preimage of Z in D'. The same argument as above shows that $\mathrm{H}^0_{\mathrm{dR}}(D'(*Z'),\mathcal{F}')$ and $\mathrm{H}^1_{\mathrm{dR}}(D'(*Z'),\mathcal{F}')$ coincide respectively with the kernel and the cokernel of the morphism

$$\nabla' : \mathcal{F}'(D') \;\to\; \mathcal{F}'(D') \otimes_{\mathscr{O}(D')} \Omega^1_{D'}(D')\ , \tag{3.39}$$

which is nothing but the morphism (3.38) tensored by K' over K. By exactness of the tensor product, we deduce that, for $i = 0, 1$, we have a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(D(*Z),\mathcal{F}) \otimes_K K' \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(D'(*Z'),\mathcal{F}')\ . \tag{3.40}$$

Since the behavior of the other invariants of the statement with respect to scalar extension is known, it is easy to check that it is enough to prove the statement with D replaced by D', or even a connected component of D'. By choosing K' appropriately, we may ensure that all those connected components are standard open pseudo-disks and that all the points of Z' are K'-rational. Consequently, from now on, we assume that $D := \{|T| < r_D\}$ with $0 < r_D \leqslant +\infty$ and that all the point of Z are K-rational.

Since Ω^1_D and $\mathcal{F}$ are both free, we can identify the connection (3.38) with an endomorphism

$$\nabla : (\mathscr{O}_D[*Z])(D)^r \;\to\; (\mathscr{O}_D[*Z])(D)^r \tag{3.41}$$

of the form $d/dT - G(T)$, with $G(T) \in M_r((\mathscr{O}_D[*Z])(D))$. We will now study this endomorphism.

Denote by $z_1, \dots, z_n$ the points of Z. We then have

$$(\mathscr{O}_D[*Z])(D) = \mathscr{O}(D)[(T - z_1)^{-1}, \dots, (T - z_n)^{-1}] \tag{3.42}$$

and a direct sum decomposition

$$(\mathscr{O}_D[*Z])(D) = \mathscr{O}(D) \oplus \bigoplus_{i=1}^{n} (T - z_i)^{-1} K[(T - z_i)^{-1}]\ . \tag{3.43}$$

We will now compute the generalized indexes of ∇ with respect to this decomposition (see Definition 2.1.1).

Remark 3.5.3. *We observe that one cannot deduce immediately that the index over $\mathscr{O}(D)[T^{-1}]$ is the sum of the generalized indexes because* the spaces are not Fréchet *and we cannot apply Proposition 2.1.12.*[9] *Therefore, we will provide along this proof an ad hoc argument of* comparison between the formal and convergent situations *to prove that the global index over $\mathscr{O}(D)$ is the sum of the local indexes.*

We will call generalized index of $\mathcal{F}$ at b_D the generalized index corresponding to the factor $\mathscr{O}(D)$ and denote it (with an abuse) by $\chi^{\mathrm{gen}}_{b_D}(\mathcal{F})$. Similarly, for every $i \in \{1, \dots, n\}$, we denote by b_i the germ of segment out of the point z_i, call generalized index of $\mathcal{F}$ at b_i the generalized index corresponding to the factor $(T - z_i)^{-1} K[(T - z_i)^{-1}]$ and denote it by $\chi^{\mathrm{gen}}_{b_i}(\mathcal{F})$. We explain below that we have actually already studied those generalized indexes.

First, remember that we have investigated the generalized indexes of ∇ acting on $\mathscr{F}(C_{b_D})$ with respect to a decomposition of the form (2.20). Let us consider this decomposition with $m = -1$

[9] More precisely, the problem is that, in order to apply Propositions 2.1.12 and 2.1.9, we would need to define a Fréchet topology on $\mathscr{O}(D)[\{(T - z_i)^{-1}\}_i]$ for which it is the topological sum of $\mathscr{O}(D)$ and $\oplus_i (T - z_i)^{-1} K[(T - z_i)^{-1}]$, where the latter is considered as a Fréchet space with respect to the trivial valuation on K. We were unable to find a topology with these properties.

and $n = 0$. We have $D_0 = D$ and the generalized indexes of $\mathcal{F}$ and $\mathscr{F}$ at b_D coincide, in the sense that one exists if, and only if, the other does and that, in this case, they have the same value. Indeed, the two endomorphisms of $\mathscr{O}(D)$ induced by ∇ by formula (2.3) already coincide. In particular, by Lemma 2.2.12 and Proposition 2.3.10, if K is not trivially valued, then $\mathcal{F}$ has finite generalized index at b_D if, and only if, $\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}$ is Fredholm at b_D and, in this case, we have

$$\chi^{\mathrm{gen}}_{b_D}(\mathcal{F}) \;=\; r + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}) \;=\; r + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}) + \mathrm{Irr}_{b_D}(\mathscr{F}), \tag{3.44}$$

where r is the rank of $\mathcal{F}$.

If K is trivially valued, then item i)-(a) of Section 2.6 ensures that $\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}$ is Fredholm at b_D (and that its generalized index is 0), hence that $\mathcal{F}$ has finite generalized index at b_D. By Proposition 2.3.10, (3.44) still holds.

Second, let $i \in \{1, \dots, n\}$ and denote by M_i the differential equation over $K((T - z_i))$ induced by $\mathcal{F}$ by tensoring (3.41) with $K((T - z_i))$. We see it as a differential equation on the punctured open unit disk over K endowed with the trivial valuation. The direct sum (2.20) then reads

$$K((T - z_i)) = (T - z_i)^{-1} K[(T - z_i)^{-1}] \oplus K[[T - z_i]]\;. \tag{3.45}$$

We call b_i^F the germ of segment out of z_i in the trivially valued case. By arguments similar to those above, we show that the generalized index of $\mathcal{F}$ at b_i coincides with that of M_i at b_i^F. In particular, Section 2.5 ensures that this generalized index always exist and is equal to $\mathrm{Irr}^F_{z_i}(\mathrm{M}_i) - r$ (see (2.105) and Lemma 2.2.12). By Proposition 3.4.1, we have

$$\chi^{\mathrm{gen}}_{b_i}(\mathcal{F}) \;=\; \chi^{\mathrm{abs}}_{b_i^F}(\mathrm{M}_i) - r \;=\; \mathrm{Irr}^F_{z_i}(\mathrm{M}_i) - r \;=\; \mathrm{Irr}_{z_i}(\mathscr{F}) - r\;. \tag{3.46}$$

The map $\nabla = d/dT - G(T)$ acts naturally on $(\mathscr{O}_D[*Z])(D)^r$ and on $K((T - z_i))^r$ for every $i \in \{1, \dots, n\}$. Let $N \geqslant 0$ such that the map $\prod_{i=1}^n (T - z_i)^N \cdot \nabla$ stabilizes $\mathscr{O}(D)^r$ and $K[[T - z_i]]^r$ for every $i \in \{1, \dots, n\}$. Set $P(T) := \prod_{i=1}^n (T - z_i)^N$.

Since the multiplication by $P(T)$ is invertible on both $(\mathscr{O}_D[*Z])(D)^r$ and $K((T - z_i))^r$ for every $i \in \{1, \dots, n\}$, ∇ has finite index exactly when $P(T)\nabla$ has and, in this case, we have

$$\chi((\mathscr{O}_D[*Z])(D)^r, P(T)\nabla) \;=\; \chi((\mathscr{O}_D[*Z])(D)^r, \nabla) \tag{3.47}$$

and, for each $i \in \{1, \dots, n\}$,

$$\chi(K((T - z_i))^r, P(T)\nabla) \;=\; \chi(K((T - z_i))^r, \nabla)\;. \tag{3.48}$$

We now consider the exact sequences

$$0 \to \mathscr{O}(D)^r \to (\mathscr{O}_D[*Z])(D)^r \to \frac{(\mathscr{O}_D[*Z])(D)^r}{\mathscr{O}(D)^r} \to 0 \tag{3.49}$$

and, for each $z \in Z$,

$$0 \to K[[T - z]]^r \to K((T - z))^r \to \frac{K((T - z))^r}{K[[T - z]]^r} \to 0\;. \tag{3.50}$$

Note that the operator $P(T)\nabla$ acts on all the terms of those exact sequences and that we have an isomorphism

$$\frac{(\mathscr{O}_D[*Z])(D)^r}{\mathscr{O}(D)^r} \simeq \bigoplus_{z \in Z} \frac{K((T - z))^r}{K[[T - z]]^r} \tag{3.51}$$

that is compatible with this action.

Lemma 3.5.4. *The differential equation $\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}$ is Fredholm at b_D if, and only if, the operator*

$P(T)\nabla$ has finite index on $\mathscr{O}(D)^r$. In this case, we have

$$\begin{aligned}\chi(\mathscr{O}(D)^r, P(T)\nabla) &= (1-\partial_{b_D}(P(T)))\cdot r + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}) + \mathrm{Irr}_{b_D}(\mathscr{F}) &(3.52)\\ &= (1+Nn)r + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}) + \mathrm{Irr}_{b_D}(\mathscr{F})\,. &(3.53)\end{aligned}$$

Proof. By Proposition 2.3.10 and Lemma 2.2.12, $\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}$ is Fredholm at b_D if, and only if, ∇ is and, in this case, we have

$$\chi^{\mathrm{gen}}_{b_D}(\mathscr{O}(C_{b_D})^r, \nabla) = r + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}) = r + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}) + \mathrm{Irr}_{b_D}(\mathscr{F})\,. \tag{3.54}$$

By Lemma 2.2.6, $P(T)$ is Fredholm at b_D and we have

$$\chi^{\mathrm{gen}}_{b_D}(\mathscr{O}(C_{b_D})^r, P(T)) = -\partial_{b_D}(P(T))\cdot r\,. \tag{3.55}$$

By assumption, C_{b_D} is an open pseudo-annulus in $Y = D - Z$ containing b_D. We deduce that, for each $z\in Z$, $|T-z|$ is log-affine of slope 1 along its skeleton when computed in the direction where $|T|$ increases, that is to say opposite to b_D. It follows that $\partial_{b_D}(P(T)) = -Nn$ and $\chi^{\mathrm{gen}}_{b_D}(\mathscr{O}(C_{b_D})^r, P(T)) = Nnr$.

By Lemma 2.1.11 and Proposition 2.2.10, ∇ is Fredholm at b_D if, and only if, $P(T)\nabla$ is and, in this case, we have

$$\chi^{\mathrm{gen}}_{b_D}(\mathscr{O}(C_{b_D})^r, P(T)\nabla) = (1+Nn)r + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}) + \mathrm{Irr}_{b_D}(\mathscr{F})\,. \tag{3.56}$$

Let us now compute this generalized index in another way. To do so, we consider the operator $P(T)\nabla$ acting on $\mathscr{O}(C_{b_D})^r$ and use the decomposition (2.20) with $m=-1$ and $n=0$. Notice that $\mathscr{O}(D)^r$ is stable under $P(T)\nabla$, therefore the truncation (2.3) applied to $P(T)\nabla : \mathscr{O}(C_{b_D})^r \to \mathscr{O}(C_{b_D})^r$ is equal to $P(T)\nabla$ itself acting on $\mathscr{O}(D)^r$. It follows, by Definition 2.2.3, that $P(T)\nabla$ has finite generalized index at b_D if, and only if, $P(T)\nabla$ has finite index on $\mathscr{O}(D)^r$ and that

$$\chi^{\mathrm{gen}}_{b_D}(\mathscr{O}(C_{b_D})^r, P(T)\nabla) = \chi(\mathscr{O}(D)^r, P(T)\nabla)\,. \tag{3.57}$$

If K is not trivially valued, then, by Lemma 2.2.12, $P(T)\nabla$ has finite generalized index at b_D if, and only if, it is Fredholm at b_D, and the result follows.

If K is trivially valued, then, by Corollary A.3.2, $\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}$ is Fredholm at b_D. The previous arguments show that this implies that $P(T)\nabla$ has finite index on $\mathscr{O}(D)^r$ and that (3.52) holds. The result follows. □

Lemma 3.5.5. *Let $z\in Z$. The operator $P(T)\nabla$ has finite index on $K[[T-z]]^r$ and we have*

$$\chi(K[[T-z]]^r, P(T)\nabla) = (1+N)r - \mathrm{Irr}_z(\mathscr{F})\,. \tag{3.58}$$

Proof. Let $i\in\{1,\dots,n\}$ such that $z=z_i$. We consider the operator ∇ as a connection on M_i, *i.e.* as a differential equation on the punctured open unit disk $C_i = \{0<|T-z_i|<1\}$ over K endowed with the trivial valuation. We denote by b_0^F (resp. b_1^F) the germ of segment at the open boundary out of 0 (resp. out of the Gauss point). By Section 2.5, we know that the radii of M_i are log-affine along the skeleton of C_i, that $\mathrm{M}_i^{\mathrm{Robba}}$ is Fredholm at b_0^F and b_1^F and that, for $j=0,1$, $\chi^{\mathrm{abs}}_{b_j^F}(\mathrm{M}_i^{\mathrm{Robba}}) = 0$.

Set $D_i = \{|T-z_i|<1\}$ over K endowed with the trivial valuation. We have $\mathscr{O}(C_i) = \mathscr{O}(D_i)[(T-z_i)^{-1}] = K((T-z_i))$ and $\mathscr{O}(D_i) = K[[T-z_i]]$. Hence we may now apply Lemma 3.5.4 to M_i, seen as a differential module over $D_i(*\{z_i\})$. It follows that $P(T)\nabla$ has finite index on $K[[T-z_i]]^r$ and that

$$\chi(K[[T-z_i]]^r, P(T)\nabla) = (1-\partial_{b_1^F}(P(T)))\cdot r + \mathrm{Irr}_{b_1^F}(\mathrm{M}_i) = (1-\partial_{b_1^F}(P(T)))\cdot r - \mathrm{Irr}_{b_0^F}(\mathrm{M}_i)\,. \tag{3.59}$$

Note that, on the skeleton of C, $|T-z_i|$ is log-affine of slope 1 when computed in the direction where $|T-z_i|$ increases and $|T-z_j|$ is constant, for each $j\neq i$ (i.e. P has no zeros in C). We deduce that $\partial_{b_1^F}(P(T))=-N$. Finally, by Proposition 3.4.1, we have $\mathrm{Irr}_{b_0^F}(\mathrm{M}_i)=\mathrm{Irr}_{z_i}(\mathscr{F})$ and the result follows. □

We now continue the proof of Theorem 3.5.2. By (3.48) and Section 2.5, for each $z\in Z$, we know that $P(T)\nabla$ has finite index on $K((T-z))^r$ and that

$$\chi(K((T-z))^r, P(T)\nabla)\;=\;0\,. \tag{3.60}$$

Therefore, we know that $P(T)\nabla$ has finite index on two terms of the exact sequence (3.50) and the value of the corresponding indexes. It follows from Lemma 1.4.12 that $P(T)\nabla$ has finite index on $K((T-z))^r/K[[T-z]]^r$ and that

$$\chi\Big(\frac{K((T-z))^r}{K[[T-z]]^r},P(T)\nabla\Big)\;=\;\mathrm{Irr}_z(\mathscr{F})-(1+N)r\,. \tag{3.61}$$

Now, looking at the exact sequence (3.49) and using the isomorphism (3.51), we deduce, by Lemma 1.4.12 again, that $P(T)\nabla$ has finite index on $(\mathscr{O}_D[*Z])(D)^r$ if, and only if, it has finite index on $\mathscr{O}(D)^r$. By Lemma 3.5.4, this happens exactly when $\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}$ is Fredholm at b_D. It follows that (a) and (b) are equivalent.

Let us now assume that (a) and (b) hold. By (3.47), Lemmas 1.4.12, 3.5.4 and (3.61), and by the fact that $\chi_c(D-Z)=1-n$, we have

$$\begin{aligned}\chi_{\mathrm{dR}}(D(*Z),\mathcal{F}) &= \chi((\mathscr{O}_D[*Z])(D)^r,P(T)\nabla) && (3.62)\\ &= \chi(\mathscr{O}(D)^r,P(T)\nabla)+\chi\Big(\frac{(\mathscr{O}_D[*Z])(D)^r}{\mathscr{O}(D)^r},P(T)\nabla\Big) && (3.63)\\ &= \Big[(1+Nn)r+\chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}})+\mathrm{Irr}_{b_D}(\mathscr{F})\Big]+\Big[\Big(\sum_{z\in Z}\mathrm{Irr}_z(\mathscr{F})\Big)-(1+N)nr\Big] && (3.64)\\ &= (1-n)r+\chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}})-\mathrm{Irr}_{D-Z}(\mathscr{F})\,. && (3.65)\end{aligned}$$

This proves (3.35). The equivalence between (c) and (d) follows. □

The presence of the freeness assumption on $\mathcal{F}$ in the statement of Theorem 3.5.2 is due to the fact that we cannot extend the scalars to a large field where $\mathcal{F}$ becomes free, as we did in most of the proofs until now, due to the lack of descent results in the context of differential equation with meromorphic singularities. To remove this assumption, we will need to impose conditions at more than one germ.

Corollary 3.5.6. *We maintain the notation of Setting 3.5.1. Let C_0 be a relatively compact open pseudo-annulus in D such that Z is contained in the connected component of $D-C_0$ that does not contain b_D. In particular, C_0 does not meet Z. Let $\{x,y\}$ be the relative boundary of C_0 in D and assume that y lies between x and the open boundary of D. Denote by $\{b_x,b_y\}$ the germs of segment at the open boundary of C_0 (directed as usual towards the interior of C_0).*

Assume that, for each $b\in\{b_D,b_x,b_y\}$, there exists an open sub-pseudo-annulus C_b in $D-Z$ representing b such that

i) $\mathscr{F}$ has log-affine radii along Γ_{C_b};[10]

ii) $\mathscr{F}^{\mathrm{Robba}}_{|C_b}$ is Fredholm at b.

[10]This is automatically true if $b\in\{b_x,b_y\}$ by relative compactness.

Then, $\mathcal{F}$ has finite-dimensional meromorphic cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(D(*Z),\mathcal{F})$. In this case we have

$$\chi_{\mathrm{dR}}(D(*Z),\mathcal{F}) \;=\; \chi_c(Y)\cdot \mathrm{rank}(\mathcal{F}) - \mathrm{Irr}_Y(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}})\,. \tag{3.66}$$

In particular, if $\chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}})=0$, the index formula holds:

$$\chi_{\mathrm{dR}}(D(*Z),\mathcal{F}) \;=\; \chi_c(Y)\cdot \mathrm{rank}(\mathcal{F}) - \mathrm{Irr}_Y(\mathscr{F})\,. \tag{3.67}$$

Proof. By the same argument as in the proof of Theorem 3.5.2, we may reduce to the case where D is an open disk or the affine line.

Denote by C the connected component of $D-\{x\}$ containing b_x. It is an open pseudo-annulus with open boundary $\{b_x,b_D\}$. Note that it is disjoint from Z.

Denote by D' the connected component of $D-\{y\}$ containing b_y. It is an open disk containing Z, with open boundary $b_{D'}=b_y$. Moreover, since D' is contained in a closed disk and since the ring of functions on such a disk is a PID, $\mathcal{F}_{|D'}$ is a free $\mathscr{O}(D')[*Z]$-module. Therefore, we can apply Theorem 3.5.2: the cohomology of $\mathcal{F}_{|D'}$ is finite-dimensional and one has (cf. (3.35))

$$\chi_{\mathrm{dR}}(D'(*Z),\mathcal{F}_{|D'}) \;=\; \chi_c(D'-Z)\cdot r - \mathrm{Irr}_{D'-Z}(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_{D'}}(\mathcal{F}^{\mathrm{Robba}}_{|C_{b_{D'}}})\,. \tag{3.68}$$

Let us now consider the open pseudo-annulus $C'=C\cap D'$. We have $\mathcal{F}_{|C}=\mathscr{F}_{|C}$ and the assumptions of Lemma 2.2.18 are fulfilled over C and over C'. Hence the cohomologies of $\mathcal{F}_{|C'}$ and of $\mathcal{F}_{|C}$ are finite-dimensional and one has

$$\chi_{\mathrm{dR}}(C(*Z),\mathcal{F}_{|C}) \;=\; \chi_{\mathrm{dR}}(C,\mathscr{F}_{|C}) \;=\; \chi^{\mathrm{abs}}_{b_x}(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F})\,, \tag{3.69}$$

$$\chi_{\mathrm{dR}}(C'(*Z),\mathcal{F}_{|C'}) \;=\; \chi_{\mathrm{dR}}(C',\mathscr{F}_{|C'}) \;=\; \chi^{\mathrm{abs}}_{b_x}(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_{D'}}(\mathscr{F})\,. \tag{3.70}$$

By Mayer-Vietoris, we find

$$\chi(D(*Z),\mathcal{F}) \;=\; \chi(D'(*Z),\mathcal{F}_{|D'}) + \chi(C(*Z),\mathcal{F}_{|C}) - \chi(C'(*Z),\mathcal{F}_{|C'}) \tag{3.71}$$

$$= \chi_c(D'-Z)\cdot r - \mathrm{Irr}_{D'-Z}(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_{D'}}(\mathcal{F}^{\mathrm{Robba}}_{|C_{b_{D'}}}) + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}) - \chi^{\mathrm{abs}}_{b_{D'}}(\mathscr{F}) \tag{3.72}$$

Now, since D' contains Z, one has $\chi_c(D'-Z)=\chi_c(Y)$. Moreover, by Proposition 2.3.10, one has $\chi^{\mathrm{abs}}_{b_D}(\mathscr{F})=\chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}})+\mathrm{Irr}_{b_D}(\mathscr{F})$ and analogous relations hold for $\chi^{\mathrm{abs}}_{b_x}(\mathscr{F})$ and $\chi^{\mathrm{abs}}_{b_{D'}}(\mathscr{F})$. Therefore

$$\mathrm{Irr}_{D'-Z}(\mathscr{F}) \;=\; -\sum_{z\in Z}\mathrm{Irr}_z(\mathscr{F}) - \mathrm{Irr}_{b_{D'}}(\mathscr{F}) \tag{3.73}$$

$$= \mathrm{Irr}_Y(\mathscr{F}) + \mathrm{Irr}_{b_D}(\mathscr{F}) - \mathrm{Irr}_{b_{D'}}(\mathscr{F}) \tag{3.74}$$

$$= \mathrm{Irr}_Y(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}) - \chi^{\mathrm{abs}}_{b_D}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}}) - \chi^{\mathrm{abs}}_{b_{D'}}(\mathscr{F}) + \chi^{\mathrm{abs}}_{b_{D'}}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_{D'}}})\,. \tag{3.75}$$

The claim follows. □

Corollary 3.5.7. *We maintain the setting 3.5.1. Assume that $\mathscr{F}$ satisfies $(\mathrm{Fin})^+_{b_D}$ (see Definition* **??***) or that*

i) $\mathscr{F}$ has log-affine radii along the germ of segment b_D at the open boundary of D;

ii) $\mathscr{F}$ is free of Liouville numbers along b_D (see Definition A.2.4).

Then, $\mathcal{F}$ has finite-dimensional meromorphic cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(D(*Z),\mathcal{F})$ and we have the index formula

$$\chi_{\mathrm{dR}}(D(*Z),\mathcal{F}) \;=\; \chi_c(Y)\cdot \mathrm{rank}(\mathcal{F}) - \mathrm{Irr}_Y(\mathscr{F})\,. \tag{3.76}$$

Proof. Condition $(\mathrm{Fin})^+_{b_D}$ clearly implies the conditions of Corollary 3.5.6. By item i) of Section 2.6, conditions i) and ii) imply $(\mathrm{Fin})^+_{b_D}$. $\square$

In the following example we analyze what happens for a rank one differential equation with an individual singularity on D.

Example 3.5.8. *Assume that $D = \{|T| < r_D\}$ as in (3.1). Let $g(T) = \sum_{n\geqslant n_0} a_n T^n \in \mathscr{O}(D)[T^{-1}]$, and consider the differential equation $\mathcal{F} : T\frac{d}{dT}(y) = g(T)y$ on $D(*0)$ with* log-affine *radii at b_D.*

If $\mathcal{F}$ is not of type Robba at the open boundary of D, then $\mathscr{F}^{\mathrm{Robba}}_{|C_{b_D}} = 0$ and the conditions of Corollary 3.5.7 are automatically satisfied. In this case, the cohomology of $\mathcal{F}$ over $D(*0)$ is finite dimensional.

Assume now that $\mathcal{F}$ is of type Robba at b_D, in particular $\mathrm{Irr}_{b_D}(\mathcal{F}) = 0$. Set $g^+(T) := \sum_{n\geqslant 1} a_n T^n$, $g^-(T) := \sum_{n=n_0}^{-1} a_n T^n$. Then $\mathcal{F}$ decomposes, over $\mathscr{O}(D)[T^{-1}]$, as

$$\mathcal{F} \;=\; \mathcal{F}^+ \otimes \mathscr{N}(a_0) \otimes \mathcal{F}^- \;, \tag{3.77}$$

where $\mathcal{F}^\pm$ and $\mathscr{N}(a_0)$ are associated with the equations $T\frac{d}{dT}(y) = g^\pm(T)y$ and $T\frac{d}{dT}(y) = a_0 \cdot y$ respectively. The concavity of the radius of convergence function on the skeleton of $D - \{0\}$ implies that $\mathcal{F}^+$, $\mathscr{N}(a_0)$, $\mathcal{F}^-$ are all of type Robba (see [Pul07, Proposition 1.1, p.501] for more details). In this case $\mathcal{F}^+$ is moreover trivial on $D(*0)$, and there exists a pseudo-annulus C_{b_D} in $D - \{0\}$ containing b_D such that $\mathcal{F}^-_{|C_{b_D}}$ is trivial. Therefore, $\chi^{\mathrm{abs}}_{b_D}(\mathcal{F}) = \chi^{\mathrm{abs}}_{b_D}(\mathscr{N}(a_0))$ and the latter equals 0 if, and only if, the residue a_0 of $g(T)$ satisfies $\mathrm{type}_{b_D}(a_0) = 1$ (cf. Definition A.1.7).

In other words if $\mathcal{F}$ is of Robba type at b_D, then its cohomology on $D(*0)$ is finite dimensional if and only if $\mathrm{type}_{b_D}(a_0) = 1$ (this is true in particular if a_0 is non-Liouville).

4. Some comparison results

In this section we list some consequence of the index Theorem 3.5.2 about the comparison between formal, meromorphic and analytic de Rham cohomologies of a differential equation over an open disk D with a meromorphic singularity at 0. Some statements generalize certain foundational results of Clark [Cla66] and Baldassarri [Bal82]. More precisely, the results of [Cla66] and Baldassarri [Bal82] hold under a "*boundary condition*" of non-Liouvilleness of the *formal* exponents at 0. The novelty of our approach is twofold : on the one hand our boundary condition consists in a systematic use of the absolute index which does not involves any Liouvilleness of the exponents, on the other hand our boundary condition arises *at the open boundary of D, not at* 0. This allows to obtain finer comparison results.

Unless specific mention, we maintain the settings of Section 3.1. We recall that $D = \{|T| < r_D\}$, b_D and b_0 are the germs of segment at the open boundary of D and out of 0 respectively, $\mathcal{F}$ is a differential equation over D possibly with meromorphic singularities at 0 and we have $\mathscr{F} = \mathcal{F}_{|D-\{0\}}$, $\mathcal{F}_0^\dagger = \mathcal{F} \otimes K(\{T\})$ and $\mathrm{M} = \mathcal{F} \otimes K((T))$.

4.1. Meromorphic vs. formal theories

In this section, we compare the meromorphic and the formal theories of differential equations.

4.1.1. Meromorphic vs. formal cohomologies.

In this section we compare the meromorphic and formal *cohomologies*. The results of this sub-section are generalizations of a result of D. Clark [Cla66, Set97].

We begin by an easy claim that does not require freeness on $\mathcal{F}$.

Lemma 4.1.1. *The natural map*

$$\mathrm{H}^i_{\mathrm{dR}}(D(*0),\mathcal{F}) \longrightarrow \mathrm{H}^i_{\mathrm{dR}}(K((T)),\mathrm{M}) \tag{4.1}$$

is injective for $i=0$ and surjective for $i=1$.

If $\mathcal{F}$ has finite-dimensional de Rham cohomology, then we have

$$\chi_{\mathrm{dR}}(D(*0),\mathcal{F}) \leqslant 0 = \chi_{\mathrm{dR}}(K((T)),\mathrm{M}) \tag{4.2}$$

and the following are equivalent:

i) $\chi_{\mathrm{dR}}(D(*0),\mathcal{F})=0$*;*

ii) for $i=0,1$, the restriction maps (4.1) are isomorphisms.

Proof. It is clear that the map $\mathrm{H}^0_{\mathrm{dR}}(D(*0),\mathcal{F}) \to \mathrm{H}^0_{\mathrm{dR}}(K((T)),\mathrm{M})$ between the vector spaces of formal and convergent solutions is injective.

On the other hand, the restriction morphism

$$(\mathscr{O}_D[*0])(D) = \mathscr{O}(D)[T^{-1}] \to K((T)) \tag{4.3}$$

has dense image. We may see M as a differential equation on an open annulus over K endowed with the trivial valuation. By Remark 1.2.4, its radii are log-affine on the skeleton. By Corollary A.3.2, $\mathrm{H}^1_{\mathrm{dR}}(K((T)),\mathrm{M})$ is finite-dimensional and this remains true after base-change to any complete extension of K. Let us consider the commutative diagram

$$\begin{array}{ccccccc} \mathcal{F}(D(*0)) & \xrightarrow{\nabla} & (\mathcal{F}\otimes_{\mathscr{O}_D}\Omega^1_D)(D(*0)) & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(D(*0),\mathcal{F}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M & \xrightarrow{\nabla} & M\otimes_{K((T))}K((T))\,dT & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(K((T)),M) & \longrightarrow & 0 \end{array} \tag{4.4}$$

It is possible to put suitable structures, namely Banachoid structures over K (see [?]), on the spaces of the bottom line such that the middle vertical map has dense image, hence the right vertical map too. Since $\mathrm{H}^1_{\mathrm{dR}}(K((T)),\mathrm{M})$ is finite-dimensional, this map is indeed surjective. We refer to the proof of [?, Lemma 4.17] for details.

Assume now that $\mathcal{F}$ has finite-dimensional de Rham cohomology. We have just proved that $\dim \mathrm{H}^1_{\mathrm{dR}}(D(*0),\mathcal{F}) \geqslant \dim \mathrm{H}^1_{\mathrm{dR}}(K((T)),\mathrm{M})$ and $\dim \mathrm{H}^0_{\mathrm{dR}}(D(*0),\mathcal{F}) \leqslant \dim \mathrm{H}^0_{\mathrm{dR}}(K((T)),\mathrm{M})$. The result now follows from Corollary A.3.2 and a computation of dimensions. □

Corollary 4.1.2. *Assume that*

i) $\mathcal{F}$ is free as $\mathscr{O}_D[*0]$-module;

ii) the radii of $\mathscr{F}$ are all log-*affine along the germ of segment b_D at the open boundary of D;*

iii) $\mathscr{F}$ is Fredholm at b_D and

$$\chi^{\mathrm{abs}}_{b_D}(\mathscr{F}) = \mathrm{Irr}_{b_D}(\mathscr{F})\,; \tag{4.5}$$

iv) $\mathrm{Irr}_{b_D}(\mathscr{F}) + \mathrm{Irr}_{b_0}(\mathscr{F}) = 0$.

Then $\mathcal{F}$ has finite-dimensional de Rham cohomology, we have $\chi_{\mathrm{dR}}(D(*0),\mathcal{F})=0$ and the restriction maps (4.1) are isomorphisms.

Proof. It is a direct consequence of Theorem 3.5.2 and Lemma 4.1.1. □

Remark 4.1.3. *The conclusions of Corollary 4.1.2 remain the same if we replace the conditions i), ii), iii) by the conditions i),ii),iii),iv) of Corollary 3.5.6, or alternatively by the conditions i),ii) of Corollary 3.5.7, where of course $Z=\{0\}$.*

Remark 4.1.4. *A direct interesting consequence of the above claim asserts, in down to earth terms, that if $Y' = G(T)Y$, with $G \in M_r(\mathscr{O}(D)[T^{-1}])$, is a differential equation satisfying the assumptions of Corollary 4.1.2, or more simply condition i) of Lemma 4.1.1, then any formal solution $Y \in K((T))^r$ is actually convergent outside 0 and lies in $\mathscr{O}(D)[T^{-1}]^r \subseteq K((T))^r$.*

Remark 4.1.5. *The assumptions of Corollary 4.1.2 do not imply that the radii of $\mathscr{F}$ are all log-affine along the whole Γ_C. In particular, the largest disk on which the maps (4.1) are isomorphisms may be bigger than the largest disk D such that $\mathscr{F}$ has log-affine radii on $\Gamma_{D-\{0\}}$. For instance, if $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are rank one meromorphic differential equations on $D(*0)$ having log-affine radii along Γ_C and if the radii of $\mathcal{F}_1$ and $\mathcal{F}_2$ cross at one point $x \in \Gamma_C$, then $\mathrm{Irr}_C(\mathcal{F}) = 0$, but the radii of $\mathcal{F}$ are not log-affine.*

Remark 4.1.6. *Let D' be a sub-disk of D. Assume that the restrictions maps (4.1) are isomorphisms for D and D' and $i = 0, 1$. Then, for $i = 0, 1$, the restriction maps*

$$\mathrm{H}^i_{\mathrm{dR}}(D(*0), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(D'(*0), \mathcal{F}_{|D'}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(K((T)), \mathrm{M}) \tag{4.6}$$

are isomorphisms too.

Notice that if the radius of D' is small enough, then items i), ii) and iv) of Corollary 4.1.2 are automatically satisfied by Lemma 3.2.3.

Remark 4.1.7. *Assume that K is trivially valued. If the radius r_D of D satisfies $r_D \leqslant 1$, then $\mathscr{O}(D)[T^{-1}] = K((T))$, therefore the above lemmas are trivially true. However, if $r_D > 1$, then one has $\mathrm{Irr}_C(\mathscr{F}) = 0$ if and only if $\mathscr{F}$ has log-affine radii along Γ_C and $\mathscr{F} = \mathscr{F}^{\mathrm{Robba}}$. Indeed, if $x_{0,1}$ denotes the point at the boundary of the annulus $\{|T| = 1\}$, then all the radii of $\mathscr{F}$ are solvable at $x_{0,1}$ (cf. Section 2.5). Therefore, if $r_D > 1$, the above lemmas only involve Robba modules.*

4.1.2. $K(\{T\})$ vs. $K((T))$. In this section, we state some consequences of the above results about differential modules over $K(\{T\})$.

The same proof as that of Lemma 4.1.1 gives the following result.

Lemma 4.1.8. *The natural map*

$$\mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^\dagger) \to \mathrm{H}^i_{\mathrm{dR}}(K((T)), \mathrm{M}) \tag{4.7}$$

is injective for $i = 0$ and surjective for $i = 1$. In particular, if the cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^\dagger)$ are finite-dimensional, then the following conditions are equivalent:

i) $\chi_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^\dagger) = 0$;

ii) for $i = 0, 1$, the restriction maps

$$\mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^\dagger) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(K((T)), \mathrm{M}) \tag{4.8}$$

are isomorphisms. □

Lemma 4.1.9. *For $i = 0, 1$, we have a natural isomorphism of K-vector spaces*

$$\mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^\dagger) \xrightarrow{\sim} \varinjlim_{D'} \mathrm{H}^i_{\mathrm{dR}}(D'(*0), \mathcal{F}_{|D'}) , \tag{4.9}$$

where D' runs through the set of subdisks of D containing 0.

Proof. The proof is formally the same as that of Lemma 2.4.2. □

Lemma 4.1.10. *Let $D_1 \supset D_2 \supset \cdots$ be a decreasing sequence of disks such that $\bigcap_n D_n = \{0\}$. If, for each $n \geqslant 1$, the cohomology group $\mathrm{H}^i_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n})$ is finite-dimensional and $\chi_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n}) =$*

0, *then, for each* $n \geqslant 1$ *and* $i = 0, 1$, *we have isomorphisms*

$$\mathrm{H}^i_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^{\dagger}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(K((T)), \mathrm{M}) \,. \tag{4.10}$$

Proof. The claim follows from Lemma 4.1.1, Remark 4.1.6 and Lemma 4.1.9. □

Corollary 4.1.11. *Let* $D_1 \supset D_2 \supset \cdots$ *be a decreasing sequence of disks with* $\bigcap_n D_n = \{0\}$. *Let* b_n *be the germ of segment at the open boundary of* D_n. *Assume that, for* n *large enough,* $\mathscr{F}$ *is Fredholm at* b_n *and* $\chi^{\mathrm{abs}}_{b_n}(\mathscr{F}) = \mathrm{Irr}_{b_n}(\mathscr{F})$. *Then, for* $i = 0, 1$ *and* n *large enough, the space* $\mathrm{H}^i_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n})$ *is finite-dimensional and we have natural isomorphisms*

$$\mathrm{H}^i_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^{\dagger}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(K((T)), \mathrm{M}) \quad . \tag{4.11}$$

In particular, the de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^{\dagger})$ *are finite-dimensional and we have*

$$\chi_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^{\dagger}) \;=\; 0 \,. \tag{4.12}$$

Proof. For n large enough, $\mathcal{F}_{|D_n}$ is a free $\mathscr{O}(D_n)[*0]$-module and, by Lemma 3.2.3, $\mathscr{F}_{|D_n}$ has log-affine radii along $\Gamma_{D_n - \{0\}}$. By Theorem 3.5.2 and Proposition 2.3.10, the cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n})$ are finite-dimensional and we have

$$\chi_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n}) = \mathrm{Irr}_{D_n - \{0\}}(\mathscr{F}_{|D_n - \{0\}}) = 0 \,. \tag{4.13}$$

The claim then follows from Lemma 4.1.10. □

4.1.3. Descent of morphisms.

In this section, we give conditions to ensure that a morphism between differential modules over $K((T))$ comes from a morphism over $\mathscr{O}(D)[T^{-1}]$. This generalizes a result of F. Baldassarri [Bal82, Theorem 2] (cf. Section 4.1.5).

Lemma 4.1.12. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be differential equations on* $D(*0)$ *(resp. over* $K(\{T\})$*) and let*

$$\beta \;:\; \mathcal{F} \otimes K((T)) \longrightarrow \mathcal{G} \otimes K((T)) \,. \tag{4.14}$$

be a morphism of differential equations over $K((T))$. *Assume that* $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ *has finite-dimensional de Rham cohomology and satisfies*

$$\chi_{\mathrm{dR}}(D(*0), \mathrm{Hom}(\mathcal{F}, \mathcal{G})) \;=\; 0 \qquad (\textit{resp. } \chi_{\mathrm{dR}}(K(\{T\}), \mathrm{Hom}(\mathcal{F}, \mathcal{G})) \;=\; 0) \,. \tag{4.15}$$

Then, there exists a unique morphism

$$\alpha \;:\; \mathcal{F} \longrightarrow \mathcal{G} \tag{4.16}$$

such that $\beta = \alpha \otimes 1$. *Moreover,* α *is a monomorphism (resp. epimorphism, isomorphism) if, and only if, so is* β.

Proof. By Lemma 4.1.1, one has

$$\begin{aligned} \mathrm{Hom}^{\nabla}(\mathcal{F}, \mathcal{G}) \;&=\; \mathrm{H}^0_{\mathrm{dR}}(D(*0), \mathrm{Hom}(\mathcal{F}, \mathcal{G})) && (4.17)\\ &\cong\; \mathrm{H}^0_{\mathrm{dR}}(K((T)), \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \otimes K((T))) && (4.18)\\ &\cong\; \mathrm{H}^0_{\mathrm{dR}}(K((T)), \mathrm{Hom}(\mathcal{F} \otimes K((T)), \mathcal{G} \otimes K((T)))) \,. && (4.19) \end{aligned}$$

Therefore, there exists a unique $\alpha : \mathcal{F} \to \mathcal{G}$ such that $\alpha \otimes 1 = \beta$.

Denote by $\mathcal{K}$ and $\mathcal{C}$ the kernel and cokernel of α respectively. If $E \subseteq D$ is a closed disk containing 0, the terms of the sequence $0 \to \mathcal{K}_{|E} \to \mathcal{F}_{|E} \to \mathcal{G}_{|E} \to \mathcal{C}_{|E} \to 0$ are free as $\mathscr{O}_E[*0]$-modules. Therefore, β is epi or mono if, and only if, so is $\mathcal{F}_{|E} \to \mathcal{G}_{|E}$.

Now, $\mathcal{K}$ and $\mathcal{C}$ are locally free over $D(*0)$ by Proposition 3.1.2. Therefore they are zero over $E(*0)$ if and only if they are zero on the whole $D(*0)$.

The case where $\mathcal{F},\mathcal{G}$ are differential modules over $K(\{T\})$ follows analogously from Lemma 4.1.8 and the fact that the tensor product $-\otimes_{K(\{T\})} K((T))$ is fully faithful. $\square$

It follows from Corollary 4.1.2 and Remark 4.1.3 that we have the following criterion.

Corollary 4.1.13. *Let $\mathcal{F}$ and $\mathcal{G}$ be differential equations over $D(*0)$ and let $\beta : \mathcal{G}\otimes K((T)) \to \mathcal{F}\otimes K((T))$ be a morphism. Assume that the differential equation $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ satisfies the assumptions of Corollary 4.1.2 (resp. the assumptions of Remark 4.1.3). Then, there exists a morphism of differential equations $\alpha : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$ such that $\beta = \alpha\otimes 1$. Moreover, α is a monomorphism (resp. epimorphism, isomorphism) if, and only if, so is β.* $\square$

4.1.4. Descent of Turrittin-Hukuhara-Levelt decomposition. We maintain the notation of Section 3.1 where $\mathrm{M} = \mathcal{F}\otimes K((T))$ and r is its rank. A well-known result of Turrittin-Hukuhara-Levelt [Tur55] (cf. [vdPS03]) proves that there exist a natural number n, a finite extension K'/K and rank-one modules $\mathrm{N}_1,\ldots,\mathrm{N}_r$ over $K'((T^{\frac{1}{n}}))$ such that $\mathrm{M}\otimes_{K((T))} K'((T^{\frac{1}{n}}))$ is successive extension of $\mathrm{N}_1,\ldots,\mathrm{N}_r$, *i.e.* $\mathrm{N}_1,\ldots,\mathrm{N}_r$ is a Jordan-Hölder sequence for the differential module $\mathrm{M}\otimes_{K((T))} K'((T^{\frac{1}{n}}))$.

To simplify the exposition, we assume $K'=K$ and $n=1$; the general case will be analyzed in Remark 4.1.17. Assume that M admits a Jordan-Hölder sequence $\mathrm{N}_1,\ldots,\mathrm{N}_r$ formed by rank one modules. Classical computations show that M is then isomorphic to a direct sum of modules of type

$$\mathrm{N}_i\otimes \mathrm{U}_m\ , \tag{4.20}$$

where, for $m\geqslant 1$, we denote by U_m the standard m-dimensional unipotent object. This is the free differential module of rank m over $\mathscr{O}(D)[T^{-1}]$ with connection $\nabla = \nabla(T\cdot\frac{d}{dT}) : \mathrm{U}_m \to \mathrm{U}_m$ given in the basis $e_1,\ldots,e_m$ of U_m by $\nabla(e_i) = e_{i+1}$ for all $i = 1,\ldots,m$ and $\nabla(e_m) = 0$.

It is easily seen that each N_i may be represented, in a basis $n_i\in\mathrm{N}_i$, by an equation of the form $T\frac{d}{dT}(y) = g_i(T)y$, with $g_i(T)\in K[T^{-1}]$. This equation defines a differential module $\mathcal{N}_i$ over $K[T,T^{-1}]$ such that $\mathcal{N}_i\otimes K((T)) \cong \mathrm{N}_i$. Therefore, in the basis $n_i\otimes e_j$, the matrix of $\nabla(T\frac{d}{dT}) : \mathrm{N}_i\otimes\mathrm{U}_m \to \mathrm{N}_i\otimes\mathrm{U}_m$ is in the Jordan form

$$\begin{pmatrix} g_i & 1 & 0 & \cdots & 0 \\ 0 & g_i & 1 & \cdots & 0 \\ \cdots & & \cdots & & \cdots \\ 0 & \cdots & & g_i & 1 \\ 0 & \cdots & & 0 & g_i \end{pmatrix} . \tag{4.21}$$

The fact that M is direct sum of the $\mathrm{N}_i\otimes\mathrm{U}_m$ implies that whole matrix of the connection of M has such blocks along the diagonal and zeroes elsewhere. In particular, the matrix lies in $M_r(K[T,T^{-1}])$ and it defines a differential module $\mathcal{M}$ over $K[T,T^{-1}]$ such that $\mathcal{M}\otimes K((T))\cong\mathrm{M}$. The family $\mathcal{N}_1,\ldots,\mathcal{N}_r$ is a Jordan-Hölder sequence of $\mathcal{M}$ over $K[T,T^{-1}]$. It is not hard to show that, up to isomorphisms, $\mathcal{M}$ is the unique differential module over $K[T,T^{-1}]$ such that

i) $\mathcal{M}\otimes K((T))\cong\mathrm{M}$;

ii) $\mathcal{M}$ is extension of (i.e. it has a Jordan-Hölder sequence formed by) rank-one modules that are regular singular at ∞.[11]

From Lemma 4.1.12 (resp. Corollary 4.1.13) we obtain the following corollary.

[11] Recall that the isomorphism class of a rank one differential module over $\mathbb{G}_m$ defined by an equation $T\frac{d}{dT}(y) = g(T)y$, with $g(T)=\sum_i a_iT^i\in K[T,T^{-1}]$, is completely determined by the tuple $(a_i)_{i\neq 0}$ and by the class of $a_0\in K/\mathbb{Z}$. Such a module is regular at ∞ (resp. at 0) if and only if $g(T)\in K[T^{-1}]$ (resp $g(T)\in K[T]$), or equivalently if its irregularity at infinity is equal to 0 (cf. Section 3.4).

Corollary 4.1.14. *We maintain notation of section 3.1. If the assumptions of Lemma 4.1.12 (resp. Corollary 4.1.13) are fulfilled by* $\mathrm{Hom}(\mathcal{F},\mathcal{M}_{|D})$ *or by* $\mathrm{Hom}(\mathcal{M}_{|D},\mathcal{F})$*, then the formal isomorphism* $\mathcal{F}\otimes K((T))\xrightarrow{\sim}\mathcal{M}\otimes K((T))$ *descends to an isomorphism*

$$\mathcal{F}\ \xrightarrow{\ \sim\ }\ \mathcal{M}_{|D} \tag{4.22}$$

over $\mathscr{O}(D)[T^{-1}]$. □

Remark 4.1.15. *The finite dimensionality of the cohomology groups* $\mathrm{H}^i_{\mathrm{dR}}(D(*0),-)$ *and the condition* $\chi_{\mathrm{dR}}(D(*0),-)=0$ *are fulfilled by* $\mathrm{Hom}(\mathcal{F},\mathcal{M}_{|D})$ *(resp.* $\mathrm{Hom}(\mathcal{M}_{|D},\mathcal{F})$*) if for all* $i=1,\dots,r$ *they hold for* $\mathcal{F}^*\otimes(\mathcal{N}_i)_{|D}$ *(resp.* $\mathcal{F}\otimes(\mathcal{N}_i)^*_{|D}$*). Indeed,* $\mathcal{M}$ *is direct sum of modules of type* $(\mathcal{N}_i)_{|D}\otimes\mathcal{U}_m$*, where* $\mathcal{U}_m$ *is the standard* m*-dimensional unipotent object over* $D(*0)$*. Therefore* $\mathrm{Hom}(\mathcal{F},\mathcal{M}_{|D})=\mathcal{F}^*\otimes\mathcal{M}_{|D}$ *is direct sum of modules of type* $\mathcal{F}^*\otimes(\mathcal{N}_i)_{|D}\otimes\mathcal{U}_m$*. These last fit into exact sequences*

$$0\to\mathcal{F}^*\otimes(\mathcal{N}_i)_{|D}\to\mathcal{F}^*\otimes(\mathcal{N}_i)_{|D}\otimes\mathcal{U}_m\to\mathcal{F}^*\otimes(\mathcal{N}_i)_{|D}\otimes\mathcal{U}_{m-1}\to 0 \tag{4.23}$$

and the claim follows by induction.

Corollary 4.1.16. *Assume that the conditions of Corollary 4.1.11 are fulfilled by* $\mathrm{Hom}(\mathcal{F}\otimes K(\{T\}),\mathcal{M}\otimes K(\{T\}))$ *or* $\mathrm{Hom}(\mathcal{M}\otimes K(\{T\}),\mathcal{F}\otimes K(\{T\}))$*. Then the formal isomorphism* $\mathcal{F}\otimes K((T))\xrightarrow{\sim}\mathcal{M}\otimes K((T))$ *descends to an isomorphism* $\mathcal{F}\otimes K(\{T\})\xrightarrow{\sim}\mathcal{M}\otimes K(\{T\})$*.* □

Remark 4.1.17. *At the beginning of this section we have supposed* $K'=K$ *and* $n=1$*. We now drop this assumption and place ourself in the general situation where* $\mathcal{M}$ *is defined after pull-back, that is over the ring* $K'[T^{\frac{1}{n}},T^{-\frac{1}{n}}]$*. The aim of this remark is to recall that* $\mathcal{M}$ *descends to a differential module over* $K[T,T^{-1}]$ *and to prove that the isomorphism obtained in Corollary 4.1.14 descends to an isomorphism over* D*.*

Without loss of generality we can assume that K'/K *is Galois. It is a classical result of N. Katz [Kat87a] that* $\mathcal{M}$ *descends into a differential module over* $K[T,T^{-1}]$*, denoted by* $\mathrm{Can}(\mathrm{M})$*, which is defined as the fixed points of* $\mathcal{M}$ *by the action of the Galois group* $\mathrm{Gal}(K'((T^{1/n}))/K((T)))$ *(this last group is implicitly identified with the Galois group of the Galois covering* $[n]:\mathbb{G}_{m,K'}\to\mathbb{G}_{m,K}$ *given by the multiplication by* n*). By Katz's construction, the isomorphism*

$$\beta'\ :\ \mathrm{M}\otimes_{K((T))}K'((T^{1/n}))\ \xrightarrow{\ \sim\ }\ \mathcal{M}\otimes_{K'[T^{\frac{1}{n}},T^{-\frac{1}{n}}]}K'((T^{1/n})) \tag{4.24}$$

descends to an isomorphism

$$\beta\ :\ \mathrm{M}\ \xrightarrow{\ \sim\ }\ \mathrm{Can}(\mathrm{M})\otimes_{K[T,T^{-1}]}K((T))\ . \tag{4.25}$$

More generally, Can *is a fully faithful functor associating to a differential module over* $K((T))$ *a differential module over* $K[T,T^{-1}]$*. This functor is called Katz's* canonical extension*.*

Now, denote by D' *the inverse image of* D *by* $[n]^{\mathrm{an}}$*. It is an open pseudo-disk centered at* 0 *too. We claim that the isomorphism* $\alpha':\mathcal{F}\otimes_{\mathscr{O}(D)[T^{-1}]}\mathscr{O}(D')[T^{-1/n}]\xrightarrow{\ \sim\ }\mathcal{M}_{|D'}$ *obtained in Corollary 4.1.14 descends to an isomorphism*

$$\alpha\ :\ \mathcal{F}\ \xrightarrow{\ \sim\ }\ \mathrm{Can}(\mathrm{M})_{|D}\ . \tag{4.26}$$

Indeed, it is enough to prove that α' *commutes with the action of the Galois group, but this follows from the fact that its restriction* $\alpha\otimes 1:\mathcal{F}\otimes K'((T^{1/n}))\xrightarrow{\sim}\mathcal{M}\otimes K'((T^{1/n}))$ *coincides with* β' *(by the identification* $\mathcal{F}\otimes K((T))=\mathrm{M}$*). This shows that* α' *descends.*

Finally we notice that, by Corollary 2.2.15, the assumptions of Lemma 4.1.12 or Corollary 4.1.13 are invariants by pull-back. Therefore those assumptions can be done directly on $\mathrm{Hom}(\mathcal{F},\mathrm{Can}(\mathrm{M})_{|D})$ *or* $\mathrm{Hom}(\mathrm{Can}(\mathrm{M})_{|D},\mathcal{F})$*.*

We resume the above Remark in the following

Corollary 4.1.18. *We maintain notation of section 3.1. If the assumptions of Lemma 4.1.12 (resp. Corollary 4.1.13) are fulfilled by* $\mathrm{Hom}(\mathcal{F},\mathrm{Can}(\mathrm{M})_{|D})$ *or by* $\mathrm{Hom}(\mathrm{Can}(\mathrm{M})_{|D},\mathcal{F})$*, then the formal isomorphism* $\mathcal{F}\otimes K((T))\xrightarrow{\sim}\mathrm{Can}(\mathrm{M})\otimes K((T))$ *descends to an isomorphism*

$$\mathcal{F}\ \xrightarrow{\ \sim\ }\ \mathrm{Can}(\mathrm{M})_{|D} \tag{4.27}$$

over $\mathscr{O}(D)[T^{-1}]$. □

4.1.5. Relations with Baldassarri's theorem. In this section we discuss the relations of Corollary 4.1.14 with Baldassarri's theorem [Bal82, Theorem 2]. We begin by recalling the classical notion of formal exponents, and state Baldassarri's theorem.

For $f=\sum_i a_iT^i$, we denote by $\mathrm{res}_0(f):=a_0$ the residue of the differential form $f\cdot\frac{dT}{T}$ and by $v_T(f)=\min\{i,\ a_i\neq 0\}$ its T-adic valuation.

The classical definition of the exponent of a formal differential module M over $K((T))$ involves the *indicial polynomial*. However, if the rank one modules $\mathrm{N}_1,\ldots,\mathrm{N}_r$ over $K((T))$ are given and if e_i is the class modulo $\mathbb{Z}$ of $\mathrm{res}_0(g_i)$, then the *formal exponent* of M is the multi-set $\{e_1,\ldots,e_r\}$ of elements of $K/\mathbb{Z}$.[12] An easy computation shows that the formal exponent of End(M) is the multi-set

$$\{e_i-e_j\}_{i,j=1,\ldots,r}\ . \tag{4.28}$$

Theorem [Bal82, Theorem 2] asserts that if the residual field of K has positive characteristic[13] and if the formal exponent of End(M) does not contain any Liouville number, then the isomorphism $\mathcal{F}\otimes K((T))\xrightarrow{\sim}\mathcal{M}\otimes K((T))$ descends to an isomorphism $\mathcal{F}\otimes K(\{T\})\xrightarrow{\sim}\mathcal{M}\otimes K(\{T\})$ over $K(\{T\})$.

We now discuss the relation with Corollary 4.1.14. The conclusion of this last is stronger than the result of Baldassarri since it provides more precision about the disk of existence D of the isomorphism $\mathcal{F}\xrightarrow{\sim}\mathcal{M}_{|D}$. Moreover, in Lemma 4.1.21 below, we show that if the characteristic of the residual field of K is positive, the assumptions of Baldassarri's theorem imply those of Corollary 4.1.16. Note however that the proof of Lemma 4.1.21 itself uses Baldassarri's theorem.

Remark 4.1.19. *The assumptions of Corollary 4.1.14 involve* $\mathrm{Hom}(\mathcal{F},\mathcal{M}_{|D})$ *while those of [Bal82, Theorem 2] involve* End(M)*. We now discuss this (apparent) discrepancy of assumptions.*

If $\mathcal{F}$ *is a differential equation on* $D(*0)$ *Baldassarri defines its formal exponent as the formal exponent of* $\mathcal{F}\otimes K((T))$ *(this last has been just defined in Remark 4.1.5). We now notice that the scalar extension* $-\otimes K((T))$ *commutes with* Hom *and we have*

$$\mathrm{End}(\mathcal{F})\otimes K((T))\ \cong\ \mathrm{End}(\mathrm{M})\ \cong\ \mathrm{Hom}(\mathcal{F},\mathcal{M}_{|D})\otimes K((T))\ . \tag{4.29}$$

Therefore, the formal exponent of $\mathrm{Hom}(\mathcal{F},\mathcal{M}_{|D})$ *coincides by definition with that of* $\mathrm{End}(\mathcal{F})$ *and we may interpret the assumption of [Bal82] as an assumption on the formal exponent of* $\mathrm{Hom}(\mathcal{F},\mathcal{M}_{|D})$.

However, in the context of Corollary 4.1.14 it seems clear that no assumptions on $\mathrm{End}(\mathcal{F})$ *can imply an isomorphism* $\mathcal{F}\xrightarrow{\sim}\mathcal{M}_{|D}$*. For instance if* $\mathcal{F}$ *has rank one, then* $\mathrm{End}(\mathcal{F})=\mathcal{F}^*\otimes\mathcal{F}$ *is automatically trivial as a differential equation over* $D(*0)$ *(hence it satisfies any condition of Lemma 4.1.12 and Corollary 4.1.13, or about the exponents), while it is easy to provide examples of rank one modules* $\mathcal{F}$ *and* $\mathcal{M}$ *that are non isomorphic as differential equations over* $D(*0)$*, but whose formal restrictions to* $K((T))$ *are isomorphic. For instance, an easy example is given by the*

[12]We notice that if we consider the trivial valuation on K and if M is seen as a differential module over the pseudo-annulus $\{0<|T|<1\}$ over K, then the formal exponent of the Robba part of M coincides with the exponent of Definition A.1.26.

[13]The main ingredient of the proof of Baldassarri is the result of Clark [Cla66] that gives, under the same assumptions as those of Baldassarri's Theorem, the equality $\mathrm{H}^0_{\mathrm{dR}}(K(\{T\}),\mathrm{End}(\mathcal{F}))\cong\mathrm{H}^0_{\mathrm{dR}}(K((T)),\mathrm{End}(\mathrm{M}))$. Up to this ingredient, the proof of Baldassarri works over general base fields. Most probably the proof of Clark may be extended to a general field too, but we did not investigated in this direction (recall that if the residual field has characteristic 0, there are no Liouville numbers, therefore no assumptions at all).

rank equations $\mathcal{M} : y' = 0$ *(trivial) and* $\mathcal{F} : y' = y$ *(exponential). If* ω *is the radius of convergence of* $\exp(T)$*, then* $\mathcal{F}$ *is trivial in the disk* $\{|T| < \omega\}$ *and non-trivial (i.e. non isomorphic to* $\mathcal{M}$*) over any larger disk.*

Remark 4.1.20. *Assume that the characteristic p of the residual field* $\widetilde{K}$ *is positive. We notice that the definition of formal exponent provided above (cf. (4.28)) does not agree with that of Christol-Mebkhout [CM97] at the germ of segment* b_0 *out of* 0 *(cf. Definition A.1.26). The reason is that this last only involves the* Robba part *of a module. More precisely, let* $C_\varepsilon = \{0 < |T| < \varepsilon\}$ *be an annulus with unspecified outer radius* ε *and let* $\mathcal{F}$ *and* $\mathcal{M}$ *be as in Section 4.1.4. If an* analytic *isomorphism* $\mathcal{F}_{|C_\varepsilon} \xrightarrow{\sim} \mathcal{M}_{|C_\varepsilon}$ *is provided over* C_ε*, then the definition of [CM97] of the exponent of* $\mathcal{F}^{\mathrm{Robba}}_{|C_\varepsilon}$ *consists in the multi-set*

$$\{e_i \ \text{ such that } \ \mathcal{N}_i = \mathcal{N}_i^{\mathrm{Robba}}\ ,\ i = 1, \ldots, r\}\ . \tag{4.30}$$

While that of $\mathrm{End}(\mathcal{F})^{\mathrm{Robba}}_{|C_\varepsilon}$ *is the multi-set*

$$\{e_i - e_j \ \text{ such that } \ \mathcal{N}_i \otimes \mathcal{N}_j^* = (\mathcal{N}_i \otimes \mathcal{N}_j^*)^{\mathrm{Robba}}\ ,\ i, j = 1, \ldots, r\}\ . \tag{4.31}$$

One proves that $\mathcal{N}_i = \mathcal{N}_i^{\mathrm{Robba}}$ *if, and only if,* $v_T(g_i) \geqslant 0$ *and* $e_i \in \mathbb{Z}_p$ *(resp.* $\mathcal{N}_i \otimes \mathcal{N}_j^* = (\mathcal{N}_i \otimes \mathcal{N}_j^*)^{\mathrm{Robba}}$ *if, and only if,* $v_T(g_i - g_j) \geqslant 0$ *and* $e_i - e_j \in \mathbb{Z}_p$*).*

In this case, the formal exponent of $\mathcal{F}$ *(resp.* $\mathrm{End}(\mathcal{F})$*) contains the Christol-Mebkhout exponent of* $\mathcal{F}$ *at* b_0 *and the inclusion may be strict. In particular, if the formal exponent does not contain any Liouville number, then so does the Christol-Mebkhout exponent (indeed Liouville numbers lie in* $\mathbb{Z}_p$*). The converse is not true in general.*

However, in the general case, if an isomorphism $\mathcal{F}_{|C_\varepsilon} \xrightarrow{\sim} \mathcal{M}_{|C_\varepsilon}$ *is not given (for instance because we do not assume any condition on the exponents nor about the indexes), the definition of the Christol-Mebkhout exponent is more involved, and the relation with the classical definition (4.28) is not clear to us.*[14]

We now prove that the assumptions of [Bal82, Theorem 2] imply those of Corollary 4.1.16 for the differential equation $\mathrm{Hom}(\mathcal{F}, \mathcal{M}_{|D}) \otimes K(\{T\})$.

Lemma 4.1.21. *We maintain the assumptions established all along Section 4.1.4. Assume that the residual field of* K *has positive characteristic and consider the following properties*

i) the formal exponent of $\mathrm{Hom}(\mathcal{F}, \mathcal{M}_{|D})$ *(i.e. the formal exponent of* $\mathrm{End}(\mathrm{M})$*, cf. Remark (4.1.19)) has no Liouville numbers;*

ii) the Christol-Mebkhout exponent of $\mathrm{Hom}(\mathcal{F}, \mathcal{M}_{|D})$ *at the germ of segment* b_0 *out of* 0 *(cf. Definition A.1.26) has no Liouville numbers.*

Then, the first property implies the second one. In this case $\mathrm{Hom}(\mathcal{F}, \mathcal{M}_{|D})$ *satisfies* $(\mathrm{Fin})_{C_\varepsilon}$ *for some open pseudo-annulus* $C_\varepsilon = \{0 < |T| < \varepsilon\}$ *and* $\mathcal{F} \otimes K(\{T\}) \xrightarrow{\sim} \mathcal{M} \otimes K(\{T\})$*. In particular the assumptions of Corollary 4.1.16 and 4.1.11 are fulfilled (cf. Section 2.6).*

Proof. If i) holds, the assumptions of Baldassarri's result are fulfilled, therefore we have an isomorphism $\mathcal{F} \otimes K(\{T\}) \xrightarrow{\sim} \mathcal{M} \otimes K(\{T\})$. In particular, there exists an unspecified pseudo-annulus $C_\varepsilon = \{0 < |T| < \varepsilon\}$ together with an isomorphism $\mathcal{F}_{|C_\varepsilon} \xrightarrow{\sim} \mathcal{M}_{|C_\varepsilon}$. Now, by Remark 4.1.20, the

[14] It is known that if $\mathscr{F}$ is of type Robba and if the Christol-Mebkhout exponent of $\mathrm{End}(\mathscr{F})$ along b_0 does not contain any Liouville number, then $\mathscr{F}_{|C_\varepsilon}$ decomposes into rank one sub-quotients (cf. Theorem A.3.1). However, it is not clear that this implies that M decomposes over $K((T))$ into rank one sub-quotients too. Moreover, in the case where M decomposes over $K((T))$, it is not clear that the decomposition descends to $K(\{T\})$, and if it does it is not clear that it coincides with that of $\mathscr{F}_{|C_\varepsilon}$.

Christol-Mebkhout exponent of $\mathrm{Hom}(\mathcal{F},\mathcal{M}_{|D})$ at b_0 is included into the formal exponent (as multisets), and hence it has no Liouville numbers. Therefore ii) holds.

The last assertion follows from Section 2.6. □

4.1.6. The restriction functor. The above comparison results permit to deduce an equivalence of categories between certain sub-categories of differential equations over $D(*0)$ and $K((T))$.

Definition 4.1.22. *Consider a derivation $d : R \to R$ on a ring R. We denote by*

$$d-\mathrm{Mod}(R) \tag{4.32}$$

the category whose objects are R-modules N together with a connection $\nabla : \mathrm{N} \to \mathrm{N}$ satisfying the Leibniz rule with respect to d and whose morphisms are R-linear maps commuting with the connections. The group of morphisms between two objects M and N will be denoted by $\mathrm{Hom}^{\nabla}(\mathrm{M},\mathrm{N})$.

Definition 4.1.23. *For $\mathrm{M},\mathrm{N} \in d-\mathrm{Mod}(R)$ we denote by $\mathrm{Ext}_{d-\mathrm{Mod}(R)}(\mathrm{M},\mathrm{N})$ the Yoneda extension group (whose elements are equivalence classes of exact sequences $0 \to \mathrm{N} \to \mathrm{E} \to \mathrm{M} \to 0$ in $d-\mathrm{Mod}(R)$). If the category $d-\mathrm{Mod}(R)$ is clear, we simply write $\mathrm{Ext}(\mathrm{M},\mathrm{N})$.*

Let $\mathcal{C}$ be a sub-category of $d-\mathrm{Mod}(R)$. We say that $\mathcal{C}$ is stable by extensions if for each exact sequence $0 \to \mathrm{M} \to \mathrm{E} \to \mathrm{N} \to 0$ in $d-\mathrm{Mod}(R)$, where M and N belong to $\mathcal{C}$, the middle term E belongs to $\mathcal{C}$ too.

The boundary conditions (as well as the Liouville conditions on the exponents) are not stable by tensor product nor by internal Hom. Hence, we consider a full sub-category $\mathcal{C}$ of $d-\mathrm{Mod}(\mathscr{O}(D)[T^{-1}])$ such that for all $\mathcal{F},\mathcal{G} \in \mathcal{C}$ the differential equation $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ satisfies the following properties:

i) $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ is free;

ii) $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ has finite-dimensional de Rham cohomology;

iii) $\chi_{\mathrm{dR}}(D(*0),\mathrm{Hom}(\mathcal{F},\mathcal{G})) = 0$.

Corollary 4.1.24. *The restriction functor*

$$Res^{D(*0)}_{K((T))} \;:\; \mathcal{C} \;\longrightarrow\; d-\mathrm{Mod}(K((T)))\;. \tag{4.33}$$

is fully faithful. Moreover, for $\mathcal{F},\mathcal{G} \in \mathcal{C}$, if we denote by $F = \mathcal{F} \otimes K((T))$ and $G = \mathcal{G} \otimes K((T))$ their images in $d-\mathrm{Mod}(K((T)))$, we have an isomorphism of Yoneda extension groups

$$\mathrm{Ext}_{d-\mathrm{Mod}(\mathscr{O}(D)[T^{-1}])}(\mathcal{F},\mathcal{G}) \;\xrightarrow{\sim}\; \mathrm{Ext}_{d-\mathrm{Mod}(K((T)))}(F,G)\;. \tag{4.34}$$

In particular, if $\mathcal{C}$ is stable by extensions in $d-\mathrm{Mod}(\mathscr{O}(D)[T^{-1}])$, then so is its essential image in $d-\mathrm{Mod}(K((T)))$.

Proof. The full-faithfullness follows from Lemma 4.1.1 applied to the differential equation $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ since

$$\mathrm{Hom}^{\nabla}(\mathcal{F},\mathcal{G}) \;=\; \mathrm{H}^0_{\mathrm{dR}}(\mathrm{Hom}(\mathcal{F},\mathcal{G})) \;=\; \mathrm{H}^0_{\mathrm{dR}}(\mathrm{Hom}(F,G)) \;=\; \mathrm{Hom}^{\nabla}(F,G)\;. \tag{4.35}$$

Let A be one of the rings $\mathscr{O}(D)[T^{-1}]$ and $K((T))$ and let $\mathrm{M},\mathrm{N} \in d-\mathrm{Mod}(A)$. It is a general fact that, when M and N are projective, we have isomorphisms $\mathrm{Ext}(\mathrm{M},\mathrm{N}) \cong \mathrm{H}^1_{\mathrm{dR}}(A,\mathrm{M}^*\otimes\mathrm{N}) \cong \mathrm{H}^1_{\mathrm{dR}}(A,\mathrm{Hom}(\mathrm{M},\mathrm{N}))$ (cf. [Ked10, Lemma 5.3.3 and Remark 5.3.4]). The isomorphism (4.34) then follows from Proposition 3.1.2 and Lemma 4.1.1. □

We now state a variant of the above corollary for the field $K(\{T\})$. It is clear that the existence of a sequence of disks as in Corollary 4.1.11 only depends on $\mathcal{F}^{\dagger}_0$, and not on the particular equation $\mathcal{F}$ defined over some disk D centered at 0 satisfying $\mathcal{F}^{\dagger}_0 = \mathcal{F} \otimes K(\{T\})$. Let $\mathcal{B}$ be a full-subcategory of $d-\mathrm{Mod}(K(\{T\}))$ such that for all $\mathcal{F}_0^\dagger,\mathcal{G}_0^\dagger\in\mathcal{B}$, the equation $\mathrm{Hom}(\mathcal{F}_0^\dagger,\mathcal{G}_0^\dagger)$ has finite-dimensional cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(K(\{T\}),\mathrm{Hom}(\mathcal{F}_0^\dagger,\mathcal{G}_0^\dagger))$ and satisfies

$$\chi_{\mathrm{dR}}(K(\{T\}),\mathrm{Hom}(\mathcal{F}_0^\dagger,\mathcal{G}_0^\dagger)) \;=\; 0\;. \tag{4.36}$$

The proof of the following Corollary is similar to that of Corollary 4.1.24 using Lemma 4.1.8 instead of Lemma 4.1.1.

Corollary 4.1.25. *The scalar-extension functor*

$$\mathrm{Res}^{K(\{T\})}_{K((T))}\;:\;\mathcal{B}\;\longrightarrow\;d-\mathrm{Mod}(K((T)))\;. \tag{4.37}$$

is fully faithful. Moreover, for $\mathcal{F}_0^\dagger,\mathcal{G}_0^\dagger\in\mathcal{B}$, if we denote by F and G their images in $d-\mathrm{Mod}(K((T)))$, we have we have an isomorphis of Yoneda extension groups

$$\mathrm{Ext}_{d-\mathrm{Mod}(K(\{T\}))}(\mathcal{F}_0^\dagger,\mathcal{G}_0^\dagger)\;\xrightarrow{\;\sim\;}\;\mathrm{Ext}_{d-\mathrm{Mod}(K((T)))}(F,G)\;. \tag{4.38}$$

In particular, if $\mathcal{B}$ is stable by extensions in $d-\mathrm{Mod}(K(\{T\}))$, then so is its essential image in $d-\mathrm{Mod}(K((T)))$. □

4.2. Meromorphic vs. analytic theories

In this section we compare the meromorphic and the analytic theories of differential equations.

We maintain the notations of Section 3.1. In particular, we have $C=D-\{0\}$ and $\mathscr{F}=\mathcal{F}_{|C}$. We further consider an open sub-pseudo-annulus

$$C'\;\subseteq\;C=D-\{0\} \tag{4.39}$$

such that $\Gamma_{C'}\subseteq\Gamma_C$. We denote by $b_{0,C'},b_{1,C'}$ the germs of segment at the boundary of C'.

4.2.1. Meromorphic vs. analytic cohomologies.

Lemma 4.2.1. *The following hold:*

i) the natural map $\mathrm{H}^0_{\mathrm{dR}}(D(*0),\mathcal{F})\to\mathrm{H}^0_{\mathrm{dR}}(C',\mathscr{F}_{|C'})$ is injective;

ii) if $\mathrm{H}^1_{\mathrm{dR}}(C',\mathscr{F}_{|C'})$ is finite-dimensional, the map $\mathrm{H}^1_{\mathrm{dR}}(D(*0),\mathcal{F})\to\mathrm{H}^1_{\mathrm{dR}}(C',\mathscr{F}_{|C'})$ is surjective;

iii) if $\mathrm{H}^1_{\mathrm{dR}}(D(*0),\mathcal{F})$ and $\mathrm{H}^1_{\mathrm{dR}}(C',\mathscr{F}_{|C'})$ are finite-dimensional, then

$$\chi_{\mathrm{dR}}(D(*0),\mathcal{F})\;\leqslant\;\chi_{\mathrm{dR}}(C',\mathscr{F}_{|C'})\;. \tag{4.40}$$

Moreover, in the setting of iii), the following conditions are equivalent:

(a) $\chi_{\mathrm{dR}}(D(*0),\mathcal{F})=\chi_{\mathrm{dR}}(C',\mathscr{F})$;

(b) for $i=0,1$, the restriction maps

$$\mathrm{H}^i_{\mathrm{dR}}(D(*0),\mathcal{F})\;\xrightarrow{\;\sim\;}\;\mathrm{H}^i_{\mathrm{dR}}(C',\mathscr{F}_{|C'}) \tag{4.41}$$

are isomorphisms.

Proof. The only difficulties lie in point ii). It follows from [**?**, Lemma 4.17] (using also Corollary A.3.3 and Remark 1.2.4 when K is trivially valued).

The rest of the proof is analogous to that of Lemma 4.1.1. □

Theorem 4.2.2. *Assume that*

i) $\mathcal{F}$ is a free $\mathscr{O}_D[*0]$-module;

ii) the radii of $\mathcal{F}$ are all log-affine at the open boundary of D;

iii) $\mathscr{F}$ *is Fredholm at* $b_{0,C'}$, $b_{1,C'}$ *and* (2.92) *holds;*

iv) $\mathscr{F}$ *is Fredholm at* b_D *and satisfies* (2.63)*;*

v) $\mathrm{Irr}_{C'}(\mathscr{F}) = \mathrm{Irr}_C(\mathscr{F})$.

Then, $\mathcal{F}$ *and* $\mathscr{F}$ *have finite-dimensional de Rham cohomology on* $D(*0)$ *and* C' *respectively and, for every* $i \geqslant 0$*, the natural morphism*

$$\mathrm{H}^i_{\mathrm{dR}}(D(*0), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(C', \mathscr{F}_{|C'}) \tag{4.42}$$

is an isomorphism.

Proof. The claim follows from Theorems 3.5.2 and 2.3.14 and Lemma 4.2.1. □

In the following statement we temporarily suspend the notations of Section 3.1

Corollary 4.2.3. *Let* Y *be a quasi-smooth* K*-analytic curve and* Z *be a locally finite subset of* K*-rational points of* Y*. Let* $\mathcal{F}$ *be a differential equation on* $Y(*Z)$*. Set* $X := Y - Z$ *and* $\mathscr{F} := \mathcal{F}_{|X}$*.*

For each $z \in Z$*, let* $D_z \subseteq Y$ *be an open disk centered at* z*. We assume that for* $z \neq z'$ *one has* $D_z \cap D_{z'} = \emptyset$*. Denote by* b_{D_z} *the open boundary of* D_z *and by* b_z *the germ of segment out of* z*.*

Assume that, for all $z \in Z$*, one has*

i) $\mathcal{F}$ *is free as* $\mathscr{O}_{D_z}[*z]$*-module;*

ii) $\mathscr{F}$ *has log-affine radii at* b_{D_z}*;*

iii) $\mathscr{F}$ *is Fredholm at* b_{D_z} *and* b_z *and satisfies* (2.92).

Then, for each $i \geqslant 0$*, we have a natural isomorphism*

$$\mathrm{H}^i_{\mathrm{dR}}(Y(*Z), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F}) \,. \tag{4.43}$$

In particular this is satisfied if, for every $z \in Z$*,* $\mathscr{F}$ *verifies* $(\mathrm{Fin})_{b_z}$ *or* $(\mathrm{Fin})^+_{b_z}$*.*

Proof. Set $D := \bigcup_{z\in Z} D_z$. We have $X \cup D = Y$ and $X \cap D = \bigcup_{z\in Z} C_z$, where $C_z = D_z - \{z\}$. The Mayer-Vietoris exact sequences for the meromorphic and analytic de Rham cohomologies fit into a two-line diagram:

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(Y(*Z),\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) \oplus \mathrm{H}^i_{\mathrm{dR}}(D(*Z),\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}^i_{\mathrm{dR}}((X\cap D)(*Z),\mathcal{F}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \oplus \mathrm{H}^i_{\mathrm{dR}}(D-Z,\mathscr{F}) \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(X\cap D,\mathscr{F}) \arrow[r] & {}
\end{tikzcd} \tag{4.44}$$

By Remark 1.5.3, we have $\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) = \mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ and $\mathrm{H}^i_{\mathrm{dR}}((X\cap D)(*Z),\mathcal{F}) = \mathrm{H}^i_{\mathrm{dR}}((X\cap D),\mathscr{F})$.

We now prove that we have an isomorphism $\mathrm{H}^i_{\mathrm{dR}}(D(*Z),\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(D-Z,\mathscr{F})$. First of all, we have $\mathrm{H}^i_{\mathrm{dR}}(D(*Z),\mathcal{F}) = \oplus_{z\in Z}\mathrm{H}^i_{\mathrm{dR}}(D_z(*z),\mathcal{F})$ and $\mathrm{H}^i_{\mathrm{dR}}(D-Z,\mathcal{F}) = \oplus_{z\in Z}\mathrm{H}^i_{\mathrm{dR}}(C_z,\mathcal{F})$, therefore we are reduced to proving that, for each $z \in Z$, one has an isomorphism $\mathrm{H}^i_{\mathrm{dR}}(D_z(*z),\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(C_z,\mathscr{F})$.

Hence, we can apply Theorem 4.2.2 to obtain the desired isomorphism. The claim now follows from the five lemma. □

Remark 4.2.4. *An interesting consequence of Corollary 4.2.3 is the fact that any global* analytic *solution of* ∇ *on* X *is actually* meromorphic *on* Y *with poles in* Z*.*

Corollary 4.2.3 together with item vi) of Section 2.6 imply the following meromorphic analogue of Proposition 1.4.9.

Proposition 4.2.5. *We maintain the notations of Corollary 4.2.3. Assume that the differential equation $\mathscr{F} := \mathcal{F}_{|X}$ over X satisfies the conditions of situation 1 of Proposition 1.4.9. Then, for all i, we have $\mathrm{H}^i_{\mathrm{dR}}(Y(*Z), \mathcal{F}) = 0$.* □

Remark 4.2.6. *Let D_1 be a subdisk of D centered at 0 and set $C_1 := D_1 - \{0\}$. Assume that the assumptions of Theorem 4.2.2 hold on D and on D_1 with respect to the same C'. In this situation, it follows from (4.42) that, for all $i \geqslant 0$, the natural restrictions*

$$\mathrm{H}^i_{\mathrm{dR}}(D(*0), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(D_1(*0), \mathcal{F}_{|D_1}) \tag{4.45}$$

are isomorphisms.

4.2.2. $D(*0)$ vs. $\mathfrak{R}_b$. We maintain the notations of Section 3.1. In this section, we consider a germ of segment $b \subset \Gamma_C$ and the Robba ring $\mathfrak{R}_b$ at b.

Corollary 4.2.7. *Assume that*

i) $\mathcal{F}$ is free as $\mathscr{O}(D)[T^{-1}]$-module;

ii) $\mathcal{F}$ has finite-dimensional de Rham cohomology on $D(*0)$;

iii) there exists a sequence $C_1 \supseteq C_2 \supseteq \cdots$ of pseudo-annuli having b at their boundary such that $\bigcap_n C_n = \emptyset$ and a complete valued extension L of K with non-trivial valuation such that, for each n, the equation $(\mathcal{F}_L)_{|(C_n)_L}$ has finite-dimensional de Rham cohomology;

iv) for each n, one has $\chi_{\mathrm{dR}}(C_n, \mathcal{F}_{|C_n}) = \chi_{\mathrm{dR}}(D(*0), \mathcal{F})$.

Then, for each n, and $i = 0, 1$ we have canonical isomorphisms

$$\mathrm{H}^i_{\mathrm{dR}}(D(*0), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(C_n, \mathscr{F}_{|C_n}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{R}_b, \mathscr{F}_{|\mathfrak{R}_b}) \,. \tag{4.46}$$

If moreover $\chi_{\mathrm{dR}}(D(*0), \mathcal{F}) = 0$, then one also has isomorphisms $\mathrm{H}^i_{\mathrm{dR}}(D(*0), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(K((T)), \mathrm{M})$.

Proof. The claim follows from Lemma 4.2.1, Corollary 2.4.3. □

Remark 4.2.8. *If the radii of $\mathcal{F}$ are log-affine at b, item iii) of Corollary 4.2.7 can be relaxed by assuming $L = K$ (cf. item ii) of Corollary 2.4.3).*

Corollary 4.2.9. *Let $C_1 \supseteq C_2 \supseteq \cdots$ be a sequence of pseudo-annuli having b at their boundary such that $\bigcap_n C_n = \emptyset$. For each $n \geqslant 1$, denote by b_n the germ of segment at the open boundary of C_n that is not b.*

Assume that

i) $\mathcal{F}$ is a free $\mathscr{O}(D)[T^{-1}]$-module;

ii) the radii of $\mathcal{F}$ are log-affine at the open boundary of D;

iii) $\mathcal{F}$ is Fredholm and satisfies (2.63) at b_D;

iv) for each $n \geqslant 1$, $\mathcal{F}$ is Fredholm at b and b_n and satisfies (2.92) over C_n;

v) for each $n \geqslant 1$, one has $\mathrm{Irr}_{C_n}(\mathscr{F}) = \mathrm{Irr}_C(\mathscr{F})$.

Then, we have the isomorphisms (4.46).

In this situation we have moreover the following facts.

(a) If $\mathrm{Irr}_C(\mathscr{F}) = 0$, then the spaces in (4.46) are also isomorphic to $\mathrm{H}^i_{\mathrm{dR}}(K((T)), \mathrm{M})$.

(b) If $b = b_0$ is the germ of segment out of 0, the spaces in (4.46) are also isomorphic to $\mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^\dagger)$.

Proof. By Theorem 4.2.2, the conditions i) to v) imply the conditions of Corollary 4.2.7. The isomorphisms (4.46) then follow from it and from Corollary 2.4.3.

Let us now prove (a). Assumptions *i)*, *ii)*, and *iii)* ensure that Theorem 3.5.2 holds and we have the index formula (3.37). If $\mathrm{Irr}_C(\mathscr{F}) = 0$, then $\chi_{\mathrm{dR}}(D(*0), \mathcal{F}) = 0$ and we can apply Lemma 4.1.1. Item (a) follows.

Let us now prove point (b). Assume that $b = b_0$. By Lemma 3.2.3, the radii of $\mathscr{F}$ are log-affine on Γ_{C_n} for n big enough. It follows by *v)* that $\mathrm{Irr}_C(\mathscr{F}) = \mathrm{Irr}_{C_n}(\mathscr{F}) = 0$ for all $n \geqslant 1$. If $D_n := C_n \cup \{0\}$, one sees that the assumptions *i)* to *v)* hold as well over D_n. Lemma 4.1.10 then implies (b). □

Remark 4.2.10. *Assume that $b = b_D$ and that $\mathcal{F}$ has the property that all its radii approach 1 as x approaches the open boundary b_D[15]. Then it is not hard to prove (using the concavity along Γ_C of the partial heights of the convergence Newton polygon) that the condition $\mathrm{Irr}_C(\mathscr{F}) = 0$ implies that the radii are log-affine along Γ_C.*

In Section 3.5.2, starting from a differential module over $\mathfrak{R}_{b_D}$, we will find a lattice $\mathcal{F}$ over $\mathscr{O}(D)[T^{-1}]$. It remains an open question to know whether one can find such a lattice satisfying $\mathrm{Irr}_C(\mathcal{F}) = 0$.

The following Corollary takes into account differential equations over $K(\{T\})$.

Corollary 4.2.11. *Let $\mathcal{F}_0^\dagger$ be a differential equation over $K(\{T\})$ and let $\mathscr{F}_0^\dagger := \mathcal{F}_0^\dagger \otimes_{K(\{T\})} \mathfrak{R}_0$. Let $D_1 \supset D_2 \supset \cdots$ be a sequence of disks with $\bigcap_n D_n = \{0\}$ and let $C_n := D_n - \{0\}$. Let b_n be the germ of segment at the open boundary of D_n and b_0 be the germ of segment out of 0. Assume that $\mathcal{F}_0^\dagger$ comes from an equation $\mathcal{F}$ over $D_1(*0)$.*

Consider the following conditions.

i) There exists $n_0 \geqslant 1$ such that, for each $n \geqslant n_0$, the cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n})$ and $\mathrm{H}^i_{\mathrm{dR}}(C_n, \mathcal{F}_{|C_n})$ are finite-dimensional and one has $\chi_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n}) = \chi_{\mathrm{dR}}(C_n, \mathscr{F}_{|C_n}) = 0$.

ii) There exists $n_0' \geqslant 1$ such that, for each $n \geqslant n_0'$, $\mathcal{F}$ is Fredholm at both b_n and b_0 and it satisfies (2.63) on both germs of segments.

Then, ii) implies i). Moreover, if i) holds, then the cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^\dagger)$ and $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{R}_0, \mathscr{F}_0^\dagger)$ are finite-dimensional and, for each $n \geqslant n_0$ and $i = 0, 1$, we have isomorphisms

$$\mathrm{H}^i_{\mathrm{dR}}(K((T)), \mathrm{M}) \xleftarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(K(\{T\}), \mathcal{F}_0^\dagger) \xleftarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(D_n(*0), \mathcal{F}_{|D_n}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(C_n, \mathscr{F}_{|C_n}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{R}_0, \mathscr{F}_0^\dagger). \tag{4.47}$$

Proof. Assume that ii) holds. Then i) follows from Theorems 3.5.2 and 2.3.14 together with Lemma 3.2.3 to ensure that the radii are log-affine on the skeleta for n big enough.

Assume that i) holds. The result now follows from Lemmas 4.1.10, 4.2.1 and Corollary 2.4.3 (together with Lemma 3.2.3 again to ensure log-affinity.) □

4.2.3. Descent of morphisms. We maintain the notations of Section 3.1. Let C' be as in (4.39).

[15] This property is called *solvability* in [CM00, CM01].

Corollary 4.2.12. *Let $\mathcal{F},\mathcal{G}$ be differential equations over $D(*0)$. Let*

$$\beta \;:\; \mathcal{F}_{|C'} \to \mathcal{G}_{|C'} \tag{4.48}$$

be a morphism. Assume that

i) $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ *is free as* $\mathscr{O}_D[*0]$*-module;*

ii) $\mathrm{H}^1_{\mathrm{dR}}(D(*0),\mathrm{Hom}(\mathcal{F},\mathcal{G}))$ *and* $\mathrm{H}^1_{\mathrm{dR}}(C',\mathrm{Hom}(\mathcal{F}_{|C'},\mathcal{G}_{|C'}))$ *are finite-dimensional;*

iii) $\chi_{\mathrm{dR}}(D(*0),\mathrm{Hom}(\mathcal{F},\mathcal{G})) = \chi_{\mathrm{dR}}(C',\mathrm{Hom}(\mathcal{F}_{|C'},\mathcal{G}_{|C'}))$.

Then there exists a unique morphism $\alpha : \mathcal{F} \to \mathcal{G}$ over $D(*0)$ such that $\beta = \alpha \otimes 1$. Moreover, α is a monomorphism (resp. epimorphism, isomorphism) if, and only if, so is β.

Proof. The proof is similar to that of Lemma 4.1.12 and by using Lemma 4.2.1. □

It follows from Theorem 4.2.2 that we have the following criterion.

Corollary 4.2.13. *Let $\mathcal{F}$ and $\mathcal{G}$ be differential equations over $D(*0)$ and let $\beta : \mathcal{F}_{|C'} \to \mathcal{G}_{|C'}$ be a morphism. Assume that*

i) $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ *is a free* $\mathscr{O}_D[*0]$*-module;*

ii) the radii of $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ *are all* log-affine *at the open boundary b_D of D;*

iii) $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ *is Fredholm and satisfies (2.63) at b_D;*

iv) $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ *is Fredholm at $b_{0,C'}$ and at $b_{1,C'}$ and (2.92) holds over C';*

v) $\mathrm{Irr}_C(\mathrm{Hom}(\mathcal{F},\mathcal{G})) = \mathrm{Irr}_{C'}(\mathrm{Hom}(\mathcal{F}_{|C'},\mathcal{G}_{|C'}))$.

Then, there exists a unique morphism of differential equations $\alpha : \mathcal{F} \to \mathcal{G}$ such that $\beta = \alpha \otimes 1$. Moreover, α is a monomorphism (resp. epimorphism, isomorphism) if, and only if, so is β. □

4.2.4. The restriction functors. In this section we compare the categories of differential equations over $D(*0)$, $K(\{T\})$, $K((T))$ and $\mathfrak{R}_0$ under some conditions on the indexes.

Let D be an open disk centered at 0, $C := D - \{0\}$ and let b be a germ of segment in Γ_C.

Let $\mathcal{A}$ be a full sub-category of the category of differential equations over $D(*0)$ with the property that for all $\mathcal{F},\mathcal{G} \in \mathcal{A}$ the differential equation $\mathcal{H} := \mathrm{Hom}(\mathcal{F},\mathcal{G})$ satisfies items i), ii), iii) and iv) of Corollary 4.2.7 (with respect to an unspecified sequence of open pseudo-annuli $\{C_n\}_n$).

Denote by $\mathcal{A}'$ a full sub-category of $\mathcal{A}$ with the property that for all $\mathcal{F},\mathcal{G} \in \mathcal{A}'$ the equation $\mathcal{H}$ satisfies moreover $\chi_{\mathrm{dR}}(D(*0),\mathcal{H}) = 0$.

Corollary 4.2.14. *The restriction functor*

$$Res^{D(*0)}_{\mathfrak{R}_b} \;:\; \mathcal{A} \;\longrightarrow\; d-\mathrm{Mod}(\mathfrak{R}_b)\;. \tag{4.49}$$

is fully faithful. Moreover, for $\mathcal{F},\mathcal{G} \in \mathcal{A}$, if we denote by $\mathcal{F}_b = \mathcal{F}\otimes\mathfrak{R}_b$ and $\mathcal{G}_b = \mathcal{G}\otimes\mathfrak{R}_b$ their images in $d-\mathrm{Mod}(\mathfrak{R}_b)$, we have an isomorphism of Yoneda extension groups

$$\mathrm{Ext}_{d-\mathrm{Mod}(\mathscr{O}(D)[T^{-1}])}(\mathcal{F},\mathcal{G}) \;\xrightarrow{\sim}\; \mathrm{Ext}_{d-\mathrm{Mod}(\mathfrak{R}_b)}(\mathcal{F}_b,\mathcal{G}_b)\;. \tag{4.50}$$

In particular, if $\mathcal{A}$ is stable by extensions in $d-\mathrm{Mod}(\mathscr{O}(D)[T^{-1}])$, then so is its essential image in $d-\mathrm{Mod}(\mathfrak{R}_b)$.

Moreover the restriction functors

$$d-\mathrm{Mod}(K((T))) \;\longleftarrow\; \mathcal{A}' \;\longrightarrow\; d-\mathrm{Mod}(\mathfrak{R}_b) \tag{4.51}$$

are both fully faithful.

Proof. The proof is similar to that of Corollary 4.1.24 by using Corollary 4.2.7 instead of Lemma 4.1.1. □

Let $\mathcal{D}$ be a full sub-category of $d-\mathrm{Mod}(K(\{T\}))$ satisfying the property that if $\mathcal{F}_0^\dagger, \mathcal{G}_0^\dagger \in \mathcal{D}$, then there exists an unspecified sequence of open disks $D_1 \supset D_2 \supset \cdots$, with $\bigcap_n D_n = \{0\}$, such that if $C_n := D_n - \{0\}$ and if the differential equation $\mathrm{Hom}(\mathcal{F}_0^\dagger, \mathcal{G}_0^\dagger)$ comes from a differential equation $\mathcal{H}$ over $D_1(*0)$, then for all n the cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(D_n(*0), \mathcal{H}_{|D_n})$ and $\mathrm{H}^i_{\mathrm{dR}}(C_n, \mathcal{H}_{|C_n})$ are finite-dimensional and one has

$$\chi_{\mathrm{dR}}(D_n(*0), \mathcal{H}_{|D_n}) \;=\; \chi_{\mathrm{dR}}(C_n, \mathcal{H}_{|C_n}) \;=\; 0\,. \tag{4.52}$$

For a differential equation $\mathcal{F}_0^\dagger$ over $\mathscr{O}(D)[T^{-1}]$ we set as usual $F := \mathcal{F}_0^\dagger \otimes K((T))$, $\mathscr{F}_0^\dagger = \mathcal{F}_0^\dagger \otimes \mathfrak{R}_0$.

Corollary 4.2.15. *The scalar-extension functors*

$$d-\mathrm{Mod}(K((T))) \;\longleftarrow\; \mathcal{D} \;\longrightarrow\; d-\mathrm{Mod}(\mathfrak{R}_0)\,. \tag{4.53}$$

are fully faithful. Moreover, for $\mathcal{F}, \mathcal{G} \in \mathcal{D}$, we have an identity of Yoneda extension groups

$$\mathrm{Ext}_{d-\mathrm{Mod}(K((T)))}(F,G) \;\xleftarrow{\sim}\; \mathrm{Ext}_{d-\mathrm{Mod}(K(\{T\}))}(\mathcal{F}_0^\dagger, \mathcal{G}_0^\dagger) \;\xrightarrow{\sim}\; \mathrm{Ext}_{d-\mathrm{Mod}(\mathfrak{R}_0)}(\mathscr{F}_0^\dagger, \mathscr{F}_0^\dagger)\,. \tag{4.54}$$

Proof. The result follows from Corollary 4.2.11. □

4.3. Formal vs. convergent decompositions.

To any differential module over $K(\{T\})$, we can associate two Newton polygons: the formal one, attached to its image in $d-\mathrm{Mod}(K((T)))$, and the convergence one, attached to its image in $d-\mathrm{Mod}(\mathfrak{R}_0)$. These polygons are related by the fact that the formal one is (up to a transformation) the derivative of the convergence one (cf. Proposition 3.4.1). This does not depend on any assumption at the boundary.

In this section we show the relations between these two decompositions under some assumptions. Namely, we maintain the assumptions of Section 3.1 and assume moreover that there exists a disk D' containing 0 and an isomorphism of $\mathscr{O}(D')[T^{-1}]$-differential modules

$$\mathcal{F}_{|D'} \;\xrightarrow{\sim}\; \mathrm{Can}(\mathrm{M})_{|D'}\,, \tag{4.55}$$

where $\mathrm{Can}(\mathrm{M})$ denotes Katz's *canonical extension* of M. For instance, by Lemma 4.1.12, we may assume that $\mathrm{Hom}(\mathcal{F}_0^\dagger, \mathrm{Can}(\mathrm{M})_0^\dagger)$ has finite-dimensional de Rham cohomology over $K(\{T\})$ and that

$$\chi_{\mathrm{dR}}(K(\{T\}), \mathrm{Hom}(\mathcal{F}_0^\dagger, \mathrm{Can}(\mathrm{M})_0^\dagger)) \;=\; 0\,. \tag{4.56}$$

When K is trivially valued, we have $K((T)) = K(\{T\}) = \mathfrak{R}_0$ and $\mathrm{M} = \mathcal{F}_0^\dagger = \mathscr{F}_0^\dagger$. In this situation, by [PP13, Section 5.7], the decomposition of M by the slopes of the formal polygon coincides with its decomposition by the radii (*i.e.* by the slopes of its convergence polygon).

Since Can is a functor, the decomposition of M by the slopes of the *formal* Newton polygon gives a decomposition of $\mathcal{F}_0^\dagger$, that we will call *the formal decomposition of* $\mathcal{F}_0^\dagger$. Corollary 4.3.4 below compares the formal decomposition of $\mathcal{F}_0^\dagger$ over $K(\{T\})$ with the decomposition by the radii of $\mathscr{F}_0^\dagger$ over $\mathfrak{R}_0$ (with no assumptions on the valuation of K).

Definition 4.3.1. *We endow $\mathbb{Q} \times \mathbb{R}_{>0}$ with the lexicographic order: $(s_1, \alpha_1) \leqslant (s_2, \alpha_2)$ if $s_1 < s_2$, or $s_1 = s_2$ and $\alpha_1 \leqslant \alpha_2$.*

For all ρ let $x_{0,\rho}$ be the Berkovich point at the boundary of the disk $\{|T| \leqslant \rho\}$. The differential module $\mathcal{F}_0^\dagger$ comes from a differential module $\mathcal{F}$ over $\mathscr{O}(D)[T^{-1}]$ for some open disk D centered at 0. By Lemma 3.2.3 and [PP13, ?????citer le fait que les hauteurs partielles sont dans $\mathbb{Q}$?????], there exists $\varepsilon \in \mathbb{R}_{>0}$ and, for every $i = 1, \ldots, r = \mathrm{rank}(\mathcal{F}_0^\dagger)$, $(s_i, \alpha_i) \in \mathbb{Q} \times \mathbb{R}_{>0}$ such that the i-th radius of $\mathscr{F}_0^\dagger$ along $]0, x_{0,\varepsilon}[$ is

$$\mathcal{R}_i(\mathscr{F}_0^\dagger, x_{0,\rho}) \;=\; \alpha_i \cdot \rho^{s_i} \;, \qquad \forall\, \rho \in]0, \varepsilon[\;. \tag{4.57}$$

Notice that, since by definition one has $\mathcal{R}_1(\mathscr{F}_0^\dagger, x_{0,\rho}) \leqslant \mathcal{R}_2(\mathscr{F}_0^\dagger, x_{0,\rho}) \leqslant \cdots \leqslant \mathcal{R}_r(\mathscr{F}_0^\dagger, x_{0,\rho})$, where r is the rank of $\mathcal{F}_0^\dagger$, then we must have

$$s_1 \;\leqslant\; s_2 \;\leqslant\; \cdots \;\leqslant\; s_r \;. \tag{4.58}$$

The decomposition of $\mathscr{F}_0^\dagger$ by the radii over $\mathfrak{R}_0$ can be written as follows:

$$\mathscr{F}_0^\dagger \;=\; \bigoplus_{(s,\alpha)\in\mathbb{Q}\times\mathbb{R}_{>0}} \mathscr{F}_0^\dagger(s,\alpha) \;, \tag{4.59}$$

where all the radii of $\mathscr{F}(s,\alpha)$ are all equal to $\alpha \cdot \rho^s$ along $]0, x_{0,\varepsilon}[$.

Definition 4.3.2. *Set*

$$\mathscr{F}_0^\dagger(s) \;:=\; \bigoplus_{s'=s} \mathscr{F}_0^\dagger(s',\alpha') \;, \tag{4.60}$$

where the sum runs on all the factors of (4.59) *such that $s' = s$. The decomposition*

$$\mathscr{F}_0^\dagger \;=\; \bigoplus_{s\in\mathbb{Q}} \mathscr{F}_0^\dagger(s) \;, \tag{4.61}$$

is called decomposition of $\mathscr{F}_0^\dagger$ by the derivative of it convergence Newton polygon.

Remark 4.3.3. *Assume that K is trivially valued. Then $K((T)) = \mathfrak{R}_0$ and we have the above decomposition also for* M. *In this case, it follows from an easy computation [PP13, Section 5.7] that $\alpha_i = 1$ for all $i = 1, \ldots, r$. Hence, there is no distinction between the decomposition* (4.59) *of* M *by the radii (i.e. by the slopes of its convergence polygon) and its decomposition* (4.61) *by the derivative of its convergence Newton polygon. Both coincide with its formal decomposition, i.e. the decomposition by the slopes of its formal Newton polygon (which is indeed, up to a transformation, the derivative of the convergence Newton polygon by Proposition 3.4.1).*

Corollary 4.3.4. *The decomposition* (4.61) *of $\mathscr{F}_0^\dagger$ by the derived convergence Newton polygon coincides with the decomposition of $\mathscr{F}_0^\dagger$ induced by that of $\mathcal{F}_0^\dagger$ by the formal Newton polygon.*

Proof. The claim follows immediately from the fact that the derivative of the convergence Newton polygon of $\mathscr{F}_0^\dagger$ coincides with the formal Newton polygon of M (up to a transformation). See Proposition Proposition 3.4.1. □

Appendix A. Local Liouville conditions.

In the literature, exponents are defined only over standard annulus and differential modules that are free over $\mathscr{O}$. In this section, we extend the definition to open pseudo-annuli and germ of segments, and we allow non-freeness. Moreover, we allow differential equations with non-affine radii. We also recall and adapt to our setting some classical definitions and results that are mainly due to Christol-Mebkhout [CM97], Dwork [Dwo97], [DGS94] and Kedlaya [Ked15].

A.1. Exponents

A.1.1. Oriented pseudo-annuli

Definition A.1.1. *Let C be an open pseudo-annulus. An* orientation *of C is the datum of a germ* *of segment b in the open boundary of C. Once an orientation b is chosen, we sometimes call* upper germ *(resp.* lower germ*) of C the germ b (resp. the germ in the open boundary of C that is not b).*

The pair (C, b) is called an oriented open pseudo-annulus.

It is convenient to fix a choice of orientation when we have an annulus embedded in the affine line.

Definition A.1.2. *Fix a coordinate T on $\mathbb{A}^{1,\mathrm{an}}_K$. Let $C = \{r_1 < |T| < r_2\}$, with $0 \leqslant r_1 < r_2 \leqslant +\infty$, be a standard open pseudo-annulus. The* standard orientation *of C is the germ $b \in \partial^o C$ pointing away from ∞,* i.e. *the germ represented by $]x_{r_2}, x_{r_2-\varepsilon}[$ for ε small enough.*

Note that an isomorphism of open pseudo-annuli induces a bijection between their open boundaries.

Definition A.1.3. *Let (C, b) and (C', b') be oriented open pseudo-annuli and let $f : C \to C'$ be an isomorphism. We say that f* preserves the orientation *if it sends b to b' and that f* reverses the orientation *otherwise.*

Note that, if f is an automorphism of an open pseudo-annulus C, then it preserves the orientation for some choice of orientation if it does for the other. Therefore, we can speak about orientation preserving automorphisms without actually choosing an orientation.

Remark A.1.4. *Let (C, b) be an oriented open pseudo-annulus.*

Let C' be an open sub-pseudo-annulus of C such that $\Gamma_{C'} \subseteq \Gamma_C$. There exists a unique germ b' in the open boundary of C' pointing in the same direction as the germ b. We call it the induced orientation on C'.

Let L be a complete valued extension of K. Let C'' be a connected component of $\pi^{-1}_{L/K}(C)$. It is an open pseudo-annulus whose open boundary contains precisely one germ b'' above b. We call it the induced orientation on C''.

The following statement ensures that the isomorphism class of a Robba module is stable by infinitesimal isomorphisms. This is an example of infinitesimal deformation.

Proposition A.1.5 ([Pul16, Theorem 4.3.1])**.** *Let C be an open pseudo-annulus. Let $\mathscr{F}$ be a finite differential equation over C of Robba type (i.e. $\mathscr{F} = \mathscr{F}^{\mathrm{Robba}}$, cf. Definition 2.3.2). Let $\sigma : C \xrightarrow{\sim} C$ be a K-automorphism of C. Assume that, for each complete valued extension L of K, each connected component of $C_L - \Gamma_{C_L}$ is globally fixed by σ_L. Then, we have an isomorphism of differential equations $\sigma^*\mathscr{F} \cong \mathscr{F}$.*

Proof. By definition, such a σ is an *infinitesimal automorphism* with respect to the empty pseudo-triangulation of C in the sense of [Pul16, Definition 3.0.2]. Moreover, since $\mathscr{F} = \mathscr{F}^{\mathrm{Robba}}$, it trivially satisfies the σ-compatibility condition [Pul16, Section 4.2]. Hence, it follows from [Pul16, Theorem 4.3.1] that we have an isomorphism $\sigma^*(\mathscr{F}) \cong \mathscr{F}$. □

Remark A.1.6. *Assume that C is a standard open pseudo-annulus: $C = \{r < |T| < s\}$. Then, the condition of the proposition is satisfied if, and only if, σ is of the form $T \mapsto T + h(T)$ with $|h(x_\rho)| < \rho$ for each $\rho \in (r, s)$.*

In general, let Ω be a spherically complete valued extension of K that is algebraically closed and with value group $\mathbb{R}_+$. Let $C_1, \dots, C_n$ be the connected components of C_Ω. They are standard open annuli and the condition of the proposition is satisfied if, and only if, for each $i \in \{1, \dots, n\}$, $(\sigma_\Omega)_{|C_i}$

is of the form above.

A.1.2. Type of a number, Liouville numbers. We denote by $|.|$ the absolute value of K, and by x_ρ the point at the boundary of the disk $\{|T| \leqslant \rho\}$. We consider the open pseudo-annulus

$$C \;:=\; \{r_1 < |T| < r_2\}\;, \qquad r_1, r_2 \in [0, +\infty]\;, \qquad r_1 < r_2\;. \tag{A.1}$$

Denote by b_1 and b_2 the germs of segments at the open boundary of C represented, for ε small enough, by $]x_{r_1}, x_{r_1+\varepsilon}[$ and $]x_{r_2-\varepsilon}, x_{r_2}[$ respectively.

Definition A.1.7. *Let $e \in K$. We set*

$$\mathrm{type}_{b_2}(e, |.|) \;:=\; \liminf_{n\to+\infty} |e+n|^{1/n}\;, \tag{A.2}$$

$$\mathrm{type}_{b_1}(e, |.|) \;:=\; \liminf_{n\to+\infty} |e-n|^{1/n}\;, \tag{A.3}$$

$$\mathrm{type}(e, |.|) \;:=\; \min\Big(\,\mathrm{type}_{b_1}(e, |.|)\;,\;\mathrm{type}_{b_2}(e, |.|)\,\Big)\;. \tag{A.4}$$

For all germ of segment b in Γ_C that is oriented as b_i we set

$$\mathrm{type}_{b}(e, |.|) \;:=\; \mathrm{type}_{b_i}(e, |.|)\;. \tag{A.5}$$

We often write $\mathrm{type}_b(e)$ or $\mathrm{type}(e)$ if no confusion is possible.

Lemma A.1.8. *Let b be a germ of segment in Γ_C. The following properties hold.*

i) For all $e \in K$, one has $\mathrm{type}_{b_1}(e, |.|) = \mathrm{type}_{b_2}(-e, |.|)$, and $\mathrm{type}(e, |.|) = \mathrm{type}(-e, |.|)$.

ii) For all $e \in K$ and $n \in \mathbb{Z}$, one has $\mathrm{type}_b(e+n, |.|) = \mathrm{type}_b(e, |.|)$.

iii) If $\psi : C \xrightarrow{\sim} C$ is an isomorphism then

$$\mathrm{type}_{\psi(b)}(e) \;=\; \begin{cases} \mathrm{type}_b(e) & \text{if } \psi \text{ preserves the orientation of } C, \\ \mathrm{type}_b(-e) & \text{if } \psi \text{ reverses the orientation of } C. \end{cases} \tag{A.6}$$

iv) For all $e \in K$, one has $\mathrm{type}(e, |.|) \leqslant 1$.

v) If the restriction of $|.|$ to $\mathbb{Z}$ is the trivial absolute value, then $\mathrm{type}(e, |.|) = 1$ for all $e \in K$.

vi) If the restriction of $|.|$ to $\mathbb{Z}$ is a p-adic absolute value and if we have either $e \notin \mathbb{Z}_p$ or $e \in \mathbb{Z}_p \cap \mathbb{Q}^{\mathrm{alg}}$, then $\mathrm{type}(e, |.|) = 1$.

Proof. The evoked equalities in i), ii) and iii) follow easily from the definition.

iv) Let $e \in K$. For all $n \in \mathbb{Z}$, we have $|e \pm n|^{1/n} \leqslant \max(1, |e|)^{1/n}$. We deduce that $\mathrm{type}(e) \leqslant 1$.

v) Assume that $|.|$ is trivial on $\mathbb{Z}$. In particular, $\mathbb{Z}$ is complete, hence closed in K. If $e \in \mathbb{Z}$, then the claim is clear. If $e \notin \mathbb{Z}$, then its distance $d := \inf_{n\in\mathbb{Z}}(|e-n|)$ to $\mathbb{Z}$ is non-zero. It follows that $\mathrm{type}(e, |.|) \geqslant 1$, hence $\mathrm{type}(e, |.|) = 1$, by iv).

vi) Assume that $|.|$ is p-adic on $\mathbb{Z}$. For all $e \notin \mathbb{Z}_p$, we have $d > 0$, hence $\mathrm{type}(e) = 1$, as before. For $e \in \mathbb{Z}_p \cap \mathbb{Q}^{\mathrm{alg}}$, a proof is given in [CR94, Prop. 11.3.4] or [DGS94, Ch.VI, Prop. 1.1]. □

Definition A.1.9 (Liouville numbers)**.** *Let $e \in K$. We say that e is Liouville with respect to $|.|$ if $\mathrm{type}(e) < 1$. If $\mathrm{type}(e) = 1$, we say that e is non-Liouville with respect to $|.|$.*

Lemma A.1.8 shows that Liouville numbers only arise in a p-adic context and that they are transcendental numbers in $\mathbb{Z}_p$.

A.1.3. Modules of type $\mathscr{N}(e)$. We maintain the notation (A.1). For $e \in K$, we denote by

$$\mathscr{N}(e) \tag{A.7}$$

the rank one differential module associated with the differential equation $\frac{d}{dT}(y) = \frac{e}{T} \cdot y$ over C. Clearly, we have $\mathscr{N}(e) \otimes \mathscr{N}(e') = \mathscr{N}(e+e')$, $\mathscr{N}(e)^* = \mathscr{N}(-e)$ and $\mathscr{N}(e+n) \simeq \mathscr{N}(e)$ for all $n \in \mathbb{Z}$.

Notation A.1.10. *We set*

$$E(K) = \begin{cases} K^\circ \text{ if } \operatorname{char}(\tilde{K}) = 0; \\ \mathbb{Z}_p \text{ if } \operatorname{char}(\tilde{K}) = p > 0. \end{cases} \tag{A.8}$$

Lemma A.1.11. *Let $e \in K$. Then, the radius of convergence function $x \mapsto \mathcal{R}_1(x, \mathscr{N}(e))$ of $\mathscr{N}(e)$ is constant on C.*

It is identically equal to 1 on Γ_C (i.e. $\mathscr{N}(e) = \mathscr{N}(e)^{\mathrm{Robba}}$) if, and only if $e \in E(K)$.

Moreover, in this case, if f is a K-linear automorphism of C, we have $f^*(\mathscr{N}(e)) \simeq \mathscr{N}(e)$ if f preserves the orientation and $f^*(\mathscr{N}(e)) \simeq \mathscr{N}(-e)$ otherwise.

Proof. The Taylor solution of the equation $\frac{d}{dT}(Y) = \frac{e}{T} \cdot Y$ at the generic point t_x is $Y(T, t_x) = \sum_{s \geqslant 0} \binom{e}{s} t_x^{-s}(T - t_x)^s$, where $\binom{e}{s} = \frac{e(e-1)(e-2)\cdots(e-s+1)}{s!}$. It follows that $\mathcal{R}_1(x, \mathscr{N}(e)) = \liminf_s \binom{e}{s}^{-1/s}$, which is a constant function on C.

Now, the exact computation of the radius is estimated in [DGS94, Chapter IV, Proposition 7.3] in the case of positive residual characteristic (and it was known since Robba [Rob85]).

If the residual characteristic is 0 we may argue as follows. The absolute value on induced on $\mathbb{Z}$ by that of K is trivial. Therefore, if $|e| > 1$, then $|\binom{e}{s}| = |e|^s$ and the radius equals $|e|^{-1} < 1$. On the other hand, if $|e| \leqslant 1$, then e belongs at most to an individual unit open disk centered at an integer. Therefore, $|\binom{e}{s}|$ is eventually constant and the radius is 1.

Let us prove the last part of the statement. Let f be a K-linear automorphism of C.

Assume that f preserves the orientation. We can write $f(T) = q(T + h(T))$ with $q \in K$, $|q| = 1$, $h \in \mathscr{O}(C)$ and $|h(x_\rho)| < \rho$ for all $\rho \in]r_1, r_2[$. Therefore f is the composition of $f_1(T) = qT$ and $f_2 := f \circ f_1^{-1}$. It is easy to show directly that we have $f_1^* \mathscr{N}(e) \cong \mathscr{N}(e)$. By Remark A.1.6, the automorphism f_2 satisfies the assumptions of Proposition A.1.5 (i.e. it is an infinitesimal automorphism [Pul16, Definition 3.0.2]), hence we have $f_2^* \mathscr{N}(e) \cong \mathscr{N}(e)$. The result follows.

Now assume that f reverses the orientation. Then there exists a K-linear automorphism g of C of the form $g : T \mapsto aT^{-1}$ for some $a \in K$ such that $g \circ f$ preserves the orientation. For the previous case, we have $(g \circ f)^*(\mathscr{N}(e)) \simeq \mathscr{N}(e)$. A direct computation shows that we have $(g^{-1})^*(\mathscr{N}(e)) \simeq \mathscr{N}(-e)$ and the result follows. □

Lemma A.1.12. *Let $e \in K$ and b be a germ of segment in Γ_C. Then*

i) $\mathscr{N}(e)$ is Fredholm at b (cf. Definition 2.2.11) if and only if $\operatorname{type}_b(e) = 1$. In this case, we have $\chi_b^{\mathrm{abs}}(\mathscr{N}(e)) = 0$;

ii) $\mathscr{N}(e)$ has finite dimensional de Rham cohomology over C if, and only if, $\operatorname{type}(e) = 1$. In this case, we have $\chi_{\mathrm{dR}}(C, \mathscr{N}(e)) = 0$.

iii) One has

$$\dim \mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{N}(e)) = \begin{cases} 1 & \text{if } e \in \mathbb{Z}\,; \\ 0 & \text{if } e \notin \mathbb{Z}\,. \end{cases} \tag{A.9}$$

and

$$\dim \mathrm{H}^1_{\mathrm{dR}}(C, \mathscr{N}(e)) = \begin{cases} 1 & \text{if } e \in \mathbb{Z}\,; \\ 0 & \text{if } e \notin \mathbb{Z} \text{ and if } e \text{ is non-Liouville}\,; \\ +\infty & \text{if } e \text{ is Liouville}\,. \end{cases} \tag{A.10}$$

Proof. Items i) and ii) result from a direct computation. See [CR94, Théorème 11.3.2] or [Rob75, Section 4.19]. Item iii) follows from ii) and from straightforward computations. □

Lemma A.1.13. *Let $e, e' \in K$. Denote by $\mathrm{Hom}^{\nabla}(\mathscr{N}(e), \mathscr{N}(e'))$ the space of $\mathscr{O}(C)$-linear homomorphisms commuting with the connections and by $\mathrm{Ext}^1(\mathscr{N}(e), \mathscr{N}(e'))$ Yoneda's group of extensions (whose elements are equivalence classes of exact sequences $0 \to \mathscr{N}(e') \to E \to \mathscr{N}(e) \to 0$). Then, we have*

$$\dim \mathrm{Hom}^{\nabla}(\mathscr{N}(e), \mathscr{N}(e')) \;=\; \begin{cases} 1 & \textit{if } e' - e \in \mathbb{Z}\,; \\ 0 & \textit{if } e' - e \notin \mathbb{Z} \end{cases} \tag{A.11}$$

and

$$\dim \mathrm{Ext}^1(\mathscr{N}(e), \mathscr{N}(e')) \;=\; \begin{cases} 1 & \textit{if } \ e' - e \in \mathbb{Z}\,; \\ 0 & \textit{if } \ e' - e \notin \mathbb{Z}, \textit{ and if } e' - e \textit{ is non-Liouville}\,; \\ +\infty & \textit{if } \ e' - e \textit{ is Liouville}\,. \end{cases} \tag{A.12}$$

Proof. Since $\mathscr{N}(e - e') = \mathscr{N}(s) \otimes \mathscr{N}(e')^{\vee}$, we have classical isomorphisms

$$\mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{N}(e' - e)) \;=\; \mathrm{Hom}^{\nabla}(\mathscr{N}(e), \mathscr{N}(e'))\,; \tag{A.13}$$

$$\mathrm{H}^1_{\mathrm{dR}}(C, \mathscr{N}(e' - e)) \;=\; \mathrm{Ext}^1(\mathscr{N}(e), \mathscr{N}(e'))\,, \tag{A.14}$$

(cf. [Ked10, Lemma 5.3.3 and Remark 5.3.4]). The claim then follows from Lemma A.1.12. □

Lemma A.1.14. *Let $\mathscr{F}$ be a differential equation on C with log-affine radii along Γ_C. Set $r' :=$* $\mathrm{rank}(\mathscr{F}^{\mathrm{Robba}})$. *Assume that $\mathscr{F}^{\mathrm{Robba}}$ may be written as extension of $\mathscr{N}(e_1), \ldots, \mathscr{N}(e_{r'})$ for $e_1, \ldots, e_{r'} \in K$. Then, we have $e_1, \ldots, e_{r'} \in R(K)$.*

Moreover, if $\mathscr{F}^{\mathrm{Robba}}$ may be written as extension of $\mathscr{N}(e'_1), \ldots, \mathscr{N}(e'_{r'})$ for $e'_1, \ldots, e'_{r'} \in K$, then, there exists a permutation $\sigma \in \mathfrak{S}_{r'}$ such that, for each $i \in \{1, \ldots, r'\}$, we have $e'_{\sigma(i)} - e_i \in \mathbb{Z}$.

Proof. The first part of the lemma follows from Lemma A.1.11 and the second from Lemma A.1.13. □

A.1.4. The group of exponents. Assume that the residue field of K has positive characteristic p. Let m be a positive integer. In this setting, Christol and Mebkhout introduce in [CM97, Sections 4 and 5] a certain equivalence relation $\overset{\mathfrak{E}}{\sim}$ on $(\mathbb{Z}_p/\mathbb{Z})^m$ and define the p-adic group of exponents of rank m as

$$\mathfrak{E}_m \;:=\; (\mathbb{Z}_p/\mathbb{Z})^m / \overset{\mathfrak{E}}{\sim}\;. \tag{A.15}$$

We omit the definition of $\overset{\mathfrak{E}}{\sim}$ which is technical and not essential for our purposes.

Definition A.1.15. *Let $\mathfrak{e} \in \mathfrak{E}_m$ and let $e = (e_1, \ldots, e_m) \in \mathbb{Z}_p^m$ be a lift of $\mathfrak{e}$.*

- *If, for all $i = 1, \ldots, m$, the number e_i is non-Liouville, we say that $\mathfrak{e}$ is non-Liouville.*
- *If, for all $i, j = 1, \ldots, m$, the difference $e_i - e_j$ is non-Liouville, we say that $\mathfrak{e}$ has non-Liouville differences.*

By [CM97, Def.4.3-1, Prop. 4.3-4, Thm. 4.4-7] (see also [Ked15, Definition 3.4.18]) these properties only depend of $\mathfrak{e}$ and not of the particular choice of the lift e.

Remark A.1.16. *Unfortunately, Liouville numbers do not behave well with respect to the usual operations and do not form a group in general. In particular, there are elements of $\mathfrak{E}_m$ satisfying one point of the above definition but not both.*

Lemma A.1.17 ([CM97, Prop. 4.4-10])**.** *Assume that $\mathfrak{e} \in \mathfrak{E}_m$ has non-Liouville differences. Then* *there exists* $(\bar{e}_1, \ldots, \bar{e}_m) \in (\mathbb{Z}_p/\mathbb{Z})^m$ *such that the class of* $\mathfrak{e}$ *for* $\overset{\mathfrak{E}}{\sim}$ *in* $(\mathbb{Z}_p/\mathbb{Z})^m$ *is given by all the vectors of the form* $(\bar{e}_{\sigma(1)}, \ldots, \bar{e}_{\sigma(m)})$ *where* σ *runs through the group of permutations of* $\{1, \ldots, m\}$.

In other words, we can identify $\mathfrak{e}$ *with a multiset of* m *elements of* $\mathbb{Z}_p/\mathbb{Z}$.[16] □

Remark A.1.18. *In the situation of Lemma A.1.17, we will sometimes talk about the exponent,* i.e. *the multiset, and sometimes about the exponents (plural),* i.e. *the elements of this multiset.*

A.1.5. The exponent of a differential module of type Robba in positive residue characteristic. In this section, we assume that the residue field of K has positive characteristic p.

Let us consider the affine K-analytic line $\mathbb{A}_K^{1,\mathrm{an}}$ and fix a coordinate T on it. The definition that follows depends on it.

Let $C := \{r_1 < |T| < r_2\}$, with $0 < r_1 < r_2 < +\infty$, be an open annulus. Let $\mathscr{F}$ be a free differential equation over C such that the radii of $\mathscr{F}$ are log-affine along Γ_C. Set $r' := \mathrm{rank}(\mathscr{F}^{\mathrm{Robba}})$. In this setting, Christol and Mebkhout define the exponent $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}) \in \mathfrak{E}_{r'}$ of $\mathscr{F}^{\mathrm{Robba}}$ (see [CM97, Definition 5.3-6] or [Ked15, Definition 3.4.11]).

Remark A.1.19. *Let* $j : K \to L$ *be an isometric extension of complete valued fields. It induces a morphism* $\pi_{L/K} : \mathbb{A}_L^{1,\mathrm{an}} \to \mathbb{A}_K^{1,\mathrm{an}}$ *and we get a differential equation* $\pi_{L/K}^*(\mathscr{F})$ *over the open annulus* $\pi_{L/K}^{-1}(C)$. *It follows from the definition that we have*

$$\mathfrak{e}(\pi_{L/K}^*(\mathscr{F})^{\mathrm{Robba}}) = \mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}) . \tag{A.16}$$

More precisely, j *may be extended to a morphism* $j : \widehat{K^{\mathrm{alg}}} \to \widehat{L^{\mathrm{alg}}}$. *Using Kedlaya's definition, the invariance boils down to the fact that, if* ζ *is a root of unity in* $\widehat{K^{\mathrm{alg}}}$ *and* $A \in \mathbb{Z}_p$, *then* $j(\zeta)$ *is a root of unity and* $j(\zeta^A) = j(\zeta)^A$.

Proposition A.1.20. *Let* C *be as above and let* f *be a* K*-linear automorphism of* C. *Let* $\mathscr{F}$ *be a differential equation over* C *with log-affine radii along* Γ_C. *Then* $f^*(\mathscr{F})^{\mathrm{Robba}} = f^*(\mathscr{F}^{\mathrm{Robba}})$.

If f *preserves (resp. reverses) the orientation, then the exponents of* $(f^*\mathscr{F})^{\mathrm{Robba}}$ *coincide with (resp. the opposite of) those of* $\mathscr{F}^{\mathrm{Robba}}$.

Proof. The radii are invariant by isomorphisms, so $f^*(\mathscr{F})^{\mathrm{Robba}} = f^*(\mathscr{F}^{\mathrm{Robba}})$.

By (A.16) we can assume that K is algebraically closed, spherically complete and that $|K| = \mathbb{R}_{\geqslant 0}$. In this case $\mathscr{F}$ is free. By Remark A.1.25, we can assume that C is an annulus.

In this case, if f preserves the orientation, the equality of the exponents follows from [CM97, Proposition 5.5-4].

Assume that f reverses the orientation. Then, we can write $f = f_1 \circ f_2$, where f_2 preserves the orientation and f_1 is of the form $x \mapsto ax^{-1}$ with $a \in K^*$. Since the claim holds when f preserves the orientation, we can assume that $f = f_1$. In this case, f is an isometric automorphism of $\mathscr{O}(C)$, therefore the properties of existence and of convergence required in [Ked10, Definition 13.5.2] are transported by pull-back. The only property that changes concerns the action of the p^n-th root of unity ξ appearing in the definition. This action is given by $x \mapsto \xi \cdot x$ on C and it produces an isomorphism $\xi^* : \mathscr{F} \xrightarrow{\sim} \xi^*\mathscr{F}$ by deformation of the connection as in [Pul16]. Explicitly, one has $\xi^*(m) = \sum_{i \geqslant 0} \frac{(\xi m - m)^i}{i!} \nabla^i(m)$, for all $m \in \mathscr{F}$. Moreover, we have $(\xi^{-1})^* = (\xi^*)^{-1}$. Now, the multiplication by ξ satisfies $\xi \circ f_1 = f_1 \circ \xi^{-1}$ as an endomorphism of $\mathscr{O}(C)$. Therefore, the definition of the exponent of $f_1^*(\mathscr{F})$ involves the pull-back by $(\xi^{-1})^*$ instead that of ξ^* and this produces the effect that the exponent changes sign. The claim follows. □

[16] i.e. a set of elements of $\mathbb{Z}_p/\mathbb{Z}$ counted with multiplicities such that the sum of the multiplicities equals m.

We now extend the definition of exponent to oriented virtual open annuli.

Definition A.1.21 (Exponent of $\mathscr{F}$)**.** *Let C be an oriented virtual open annulus over K. Let $\mathscr{F}$ be a differential equation over C with* log-*affine radii along Γ_C. Set $r' := \mathrm{rank}(\mathscr{F}^{\mathrm{Robba}})$.*

Let Ω be a complete valued field extension of K that is algebraically closed, spherically complete and such that $|\Omega| = \mathbb{R}_{\geqslant 0}$. Denote by $\pi_{\Omega/K} : C_\Omega \to C$ the projection morphism. Let C' be a connected component of C_Ω and endow it with the orientation induced by that of C. It is an oriented open annulus on which $\pi^*_{\Omega/K}(\mathscr{F})$ is free. Choose an embedding φ of C into the affine line such that the image of the upper germ of C points away from ∞ and identify C with its image. We define the exponent of $\mathscr{F}^{\mathrm{Robba}}$ as

$$\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}) \ := \ \mathfrak{e}((\pi^*_{\Omega/K}(\mathscr{F})_{|C'})^{\mathrm{Robba}}) \in \mathfrak{E}_{r'} \ . \tag{A.17}$$

Remark A.1.22. *Assume that $C = \{r_1 < |T| < r_2\}$ as in the beginning of the section. If we endow C with the standard orientation (see Definition A.1.2), then, in the previous definition, one may choose φ to be the inclusion into $\mathbb{A}^{1,\mathrm{an}}_K$, hence the definition coincides with that of Christol and Mebkhout.*

Lemma A.1.23. *Definition A.1.21 does not depend on the choices of Ω, C' and φ.*

Proof. The independence of φ follows from Proposition A.1.20, so we need only consider what happens when Ω and C change.

Let (Ω_1, C'') be another choice. We can find a larger field Ω_2 with the same properties containing both Ω_1 and Ω. By Remark A.1.19, we have $\mathfrak{e}((\pi^*_{\Omega_2/K}(\mathscr{F})_{|C'_{\Omega_2}})^{\mathrm{Robba}}) = \mathfrak{e}((\pi^*_{\Omega/K}(\mathscr{F})_{|C'})^{\mathrm{Robba}})$ and $\mathfrak{e}((\pi^*_{\Omega_2/K}(\mathscr{F})_{|C''_{\Omega_2}})^{\mathrm{Robba}}) = \mathfrak{e}((\pi^*_{\Omega_1/K}(\mathscr{F})_{|C''})^{\mathrm{Robba}})$. It follows that we may assume that $\Omega_1 = \Omega$.

Let us embed $\widehat{K^{\mathrm{alg}}}$ into Ω'. The projections C'_0 and C''_0 of C' and C'' are two connected components of $\pi^{-1}_{\widehat{K^{\mathrm{alg}}}/K}(C)$. It follows that there exists a K-linear isometric automorphism of $\widehat{K^{\mathrm{alg}}}$ sending C'_0 to C''_0. By [PP15, Corollary 2.18], it extends to a K-linear isometric automorphism of Ω that sends C' to C'' and the result follows from Remark A.1.19 again, using the fact that $\sigma^*(\mathscr{F}) = \mathscr{F}$. □

Remark A.1.24. *It follows readily from the definition that the exponent is invariant under extension of scalars. More precisely, in the setting of Definition A.1.21, for each complete valued extension L of K and each connected component C' of $\pi^{-1}_{L/K}(C)$ with the induced orientation, we have*

$$\mathfrak{e}(\pi^*_{L/K}(\mathscr{F})^{\mathrm{Robba}}_{|C'}) \ = \ \mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}) \ . \tag{A.18}$$

Il also follows from Proposition A.1.20 that, if we reverse the orientation of C, then the exponent is changed into its opposite.

Lemma A.1.25. *Let C be an oriented virtual open annulus over K. Let $\mathscr{F}$ be a differential equation over C with* log-*affine radii along Γ_C. Let $C' \subseteq C$ be a virtual open annulus such that $\Gamma_{C'} \subseteq \Gamma_C$ and endow it with the induced orientation. Then, the exponent of $\mathscr{F}$ coincides with the exponent of its restriction $\mathscr{F}_{|C'}$:*

$$\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}) \ = \ \mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}_{|C'}) \ . \tag{A.19}$$

Proof. See [CM97, after Définition 5.3-6] or [Ked15, Theorem 3.4.16]. □

Definition A.1.26 (Exponent, case of pseudo-annuli and good germs of segments)**.** *Let C be an oriented pseudo-annulus and let $\mathscr{F}$ be a differential equation with log-affine radii along Γ_C. Let C'*

be a relatively compact open sub-pseudo-annulus of C such that $\Gamma_{C'} \subset \Gamma_C$ and endow it with the induced orientation. By [PP13, Lemma 1.1.28], C' is a virtual open annulus and we define the exponent of $\mathscr{F}^{\mathrm{Robba}}$ as

$$\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}) \;:=\; \mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}_{|C'}) \,. \tag{A.20}$$

By Lemma A.1.25, it does not depend on the choice of C'.

Let b be a good germ of segment in a curve X and let $\mathscr{F}$ be a differential equation on X with log-affine radii along b. Let C be an open pseudo-annulus in X whose skeleton represents b and such that $\mathscr{F}_{|C}$ has log-affine radii along Γ_C. Endow C with the orientation b. Then, we define the exponent of $\mathscr{F}^{\mathrm{Robba}}$ at b as $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}_{|C})$. By Lemma A.1.25, it does not depend on the choice of C.

Remark A.1.27. *The exponent over a pseudo-annulus or a good germ of segment is invariant under extension of scalars, in the sense of Remark A.1.24. The exponent over a pseudo-annulus is changed into its opposite when the orientation of the pseudo-annulus is reversed.*

A.1.6. The exponent of a differential module of type Robba in residue characteristic 0.

In this section, we assume that the residue field of K has characteristic 0 (this includes in particular the case where K is trivially valued).

Let us consider the affine K-analytic line $\mathbb{A}^{1,\mathrm{an}}_K$ and fix a coordinate T on it. The definition that follows depends on it. The theory of exponents will be based on the following result.

Theorem A.1.28 ([Ked15, Theorem 3.3.6])**.** *Let $C := \{r_1 < |T| < r_2\}$, with $0 \leqslant r_1 < r_2 \leqslant \infty$, be a standard open pseudo-annulus.*

Let $\mathscr{F}$ be a free differential equation over C such that the radii of $\mathscr{F}$ are log-affine *along Γ_C. Set $r' :=$ rank$(\mathscr{F}^{\mathrm{Robba}})$. Then, there exists a basis of $\mathscr{F}^{\mathrm{Robba}}$ in which the associated differential equation has the form*

$$\frac{d}{dT} - \frac{1}{T}\cdot B\,, \qquad \textit{with} \quad B \in M_{r'\times r'}(K^\circ)\,. \tag{A.21}$$

□

Remark A.1.29. *Let $e_1,\dots,e_{r'} \in K^\circ$ be the eigenvalues of the matrix B of the theorem. Let K' be a complete valued extension of K in which the matrix B can be put in Jordan form. Then, for each connected component C' of $\pi^{-1}_{K'/K}(C)$, the differential equation $\pi^*_{K'/K}(\mathscr{F})^{\mathrm{Robba}}_{|C'}$ may be written as an extension of the modules $\mathscr{N}(e_1),\dots,\mathscr{N}(e_{r'})$.*

By Lemma A.1.14, the multiset $\{e_1,\dots,e_{r'}\}$ of elements $K^\circ/\mathbb{Z}$ is independent of the choices.

Definition A.1.30 (Exponent of $\mathscr{F}$)**.** *Let C be an oriented standard open pseudo-annulus over K and let $\mathscr{F}$ be a free differential equation over C such that the radii of $\mathscr{F}$ are all* log-affine *along Γ_C. Set $r' =$ rank$(\mathscr{F}^{\mathrm{Robba}})$.*

Choose an embedding φ of C into the affine line such that the image of the upper germ of C points away from ∞ and identify C with its image. We define the exponent $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}})$ of $\mathscr{F}^{\mathrm{Robba}}$ as the finite multisubset of $\left(K^\circ/\mathbb{Z}\right)^{r'}/\mathfrak{S}_{r'}$ formed by the eigenvalues of a matrix $B \in M_{r'\times r'}(K^\circ)$ associated with $\mathscr{F}^{\mathrm{Robba}}$ by Theorem A.1.28.

By Remark A.1.29, the exponent is independent of the choice of φ.

Remark A.1.31. *It follows from the definition that the exponent is invariant under restriction and extension of scalars. More precisely, in the setting of Definition A.1.30, for each open sub-pseudo-*

annulus C' of C such that $\Gamma_{C'} \subseteq \Gamma_C$ endowed with the induced orientation, we have

$$\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}_{|C'}) \;=\; \mathfrak{e}(\mathscr{F}^{\mathrm{Robba}}) \tag{A.22}$$

and, for each complete valued extension L of K and each connected component C'' of $\pi^{-1}_{L/K}(C)$ endowed with the induced orientation, we have

$$\mathfrak{e}(\pi^{*}_{L/K}(\mathscr{F})^{\mathrm{Robba}}_{|C''}) \;=\; \mathfrak{e}(\mathscr{F}^{\mathrm{Robba}})\,. \tag{A.23}$$

Since, in residue characteristic 0, exponents do not seem to be invariant by the Galois action (compare with Remark A.1.19), we will not try to define them in a more general setting.

A.2. Liouville conditions.

We now explain what is an equation free of Liouville numbers. We firstly define this notion over open pseudo-annuli (cf. Definition A.2.1), then over good germs of segments in X on which the radii of $\mathscr{F}$ are log-affine (cf. Definition A.2.4).

We maintain the notations of the above sections.

Definition A.2.1 (Liouville conditions, case of pseudo-annuli)**.** *Let C be an oriented open pseudo-annulus. Let $\mathscr{F}$ be a differential equation on C whose radii are* log-affine *along Γ_C.*

We say that $\mathscr{F}$ is free of Liouville numbers *along Γ_C if either $\widetilde{K}$ has characteristic 0 or $\widetilde{K}$ has characteristic $p > 0$ and the following conditions hold:*

i) the exponent $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}})$ is non-Liouville;

ii) the exponent $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}})$ has non-Liouville differences.

We say that $\mathscr{F}$ is strongly free of Liouville numbers along Γ_C *if both $\mathscr{F}$ and* End$(\mathscr{F})$ *have log-affine radii along Γ_C and are free of Liouville numbers along it.*

Remark A.2.2. *Assume that K has residual characteristic 0 and that C is a standard open pseudo-annulus. In this case, the above definition states that any differential equation is free of Liouville numbers as soon as it has log-affine radii along Γ_C. This is consistent with the fact that, in this setting, there are no Liouville numbers by item v) of Lemma A.1.8.*

The following results follows readily from Lemma A.1.25 and Remark A.1.27.

Lemma A.2.3. *Let C be an oriented open pseudo-annulus. Let $\mathscr{F}$ be a differential equation on C whose radii are* log-affine *along Γ_C.*

Let $C' \subseteq C$ be an open pseudo-annulus such that $\Gamma_{C'} \subseteq \Gamma_C$ and endow it with the induced orientation. Then, $\mathscr{F}$ is free of Liouville numbers along Γ_C if, and only if, $\mathscr{F}_{|C'}$ is free of Liouville numbers along $\Gamma_{C'}$.

Let L be a complete valued extension of K and let C'' be a connected component of $\pi^{-1}_{L/K}(C)$. Endow it with the induced orientation. Then, $\mathscr{F}$ is free of Liouville numbers along Γ_C if, and only if, $\pi^{*}_{L/K}(\mathscr{F})_{|C''}$ is free of Liouville numbers along $\Gamma_{C''}$. □

We now state the principal notion introduced in this appendix: being free of Liouville numbers at a germ of segment b. Notice that we do not require log-affineness of the radii along b.

Definition A.2.4 (Liouville condition, case of good germs)**.** *Let b be a good germ of segment in X. We say that $\mathscr{F}$ is free of Liouville numbers along b if, for each open pseudo-annulus C whose skeleton represents b, there exists an open sub-pseudo-annulus $C' \subseteq C$ with $\Gamma_{C'} \subseteq \Gamma_C$ (possibly not representing b) such that $\mathscr{F}$ has log-affine radii along $\Gamma_{C'}$ and is free of Liouville numbers along it.*

We say that $\mathscr{F}$ is strongly free of Liouville numbers along b *if both* $\mathscr{F}$ *and* $\mathrm{End}(\mathscr{F})$ *are free of Liouville numbers along it.*[17]

In particular, it makes sense to say that $\mathscr{F}$ is free of Liouville numbers on b, for any germ of segment b out of a point $x \in X$.

Remark A.2.5. *In the paper we use in an essential way the freeness of Liouville numbers of the differential equation* $\mathrm{End}(\mathscr{F})$*. We notice that, even if* $\mathscr{F}$ *has log-affine radii along* Γ_C*, the same property need not hold for the differential equation* $\mathrm{End}(\mathscr{F})$*. The relation between the radii of* $\mathscr{F}$ *and those of* $\mathrm{End}(\mathscr{F}) = \mathscr{F} \otimes \mathscr{F}^*$ *is unfortunately unclear (e.g. Example* **??***).*

Notice moreover that, $\mathrm{End}(\mathscr{F})$ *always has a non-trivial submodule generated by the identity endomorphism, so if the radii of* $\mathrm{End}(\mathscr{F})$ *are log-affine along* Γ_C *we must have* $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}} \neq 0$ *(even in the case where* $\mathscr{F}^{\mathrm{Robba}} = 0$*). More precisely, if the radii of* $\mathrm{End}(\mathscr{F})$ *are log-affine along* Γ_C*, we have*

$$\mathrm{End}(\mathscr{F}^{\mathrm{Robba}}) \subseteq \mathrm{End}(\mathscr{F})^{\mathrm{Robba}} \tag{A.24}$$

and the inclusion is strict whenever $\mathscr{F} \neq \mathscr{F}^{\mathrm{Robba}}$*. Moreover applying the decomposition from Definition 2.3.2 to* $\mathscr{F}$ *and* $\mathrm{End}(\mathscr{F}) = \mathscr{F}^* \otimes \mathscr{F}$*, one sees that* $\mathrm{End}(\mathscr{F}^{\mathrm{Robba}})$ *is a non-trivial direct summand of* $\mathrm{End}(\mathscr{F})$*, and hence also of* $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$*.*

Remark A.2.6. *i) It follows from Lemma A.2.3 that the property of being free or strongly free of Liouville numbers along a good germ of segment is invariant under extension of scalars.*

ii) Definition A.2.4 generalizes Definition A.2.1 in the following sense. Let b *be a good germ of segment represented by an open pseudo-annulus* C *and assume that* $\mathscr{F}$ *is free of Liouville numbers along* b*. If the radii of* $\mathscr{F}$ *are all log-affine along* Γ_C*, then* $\mathscr{F}$ *is free of Liouville numbers all along the whole* Γ_C *by Lemma A.1.25.*

iii) We recall that $\mathscr{F}$ *is automatically free of Liouville numbers at* b *in the following cases:*

(a) b *is a germ of segment out of a point* x *on which* $\mathscr{F}$ *has spectral non solvable radii. Indeed, in that case, by continuity, the radii of* $\mathscr{F}$ *are spectral log-affine and spectral non solvable along* b*, hence* $\mathscr{F}^{\mathrm{Robba}} = 0$*.*

In this case, as observed in Remark A.2.5, the radii of $\mathrm{End}(\mathscr{F})$ *are never all spectral non-solvable and we cannot say anything about its freeness of Liouville numbers.*

(b) b *is the germ of segment at the open boundary of a virtual open disk* D *and* $\mathscr{F}$ *is a differential equation over* D *whose radii are constant along* b*. In this case, if* C *is any virtual open annulus representing* b*, the Robba part* $(\mathscr{F}_{|C})^{\mathrm{Robba}}$ *of* $\mathscr{F}_{|C}$ *is a trivial differential equation (see Lemma A.4.1) and the result follows.*

In this situation, we cannot say anything about the affineness of the radii of $\mathrm{End}(\mathscr{F})$ *nor about its freeness of Liouville numbers (compare with Corollary* **??***).*

We finally explain how to take into account the case where the equation has some meromorphic singularities. Let X be a quasi-smooth K-analytic curve, Z a locally finite set of rigid points and $\mathcal{F}$ a differential equation on X with meromorphic singularities at Z. Set $Y := X - Z$ and $\mathscr{F} := \mathcal{F}_{|Y}$.

A good germ of segment b of X can be represented by an open pseudo-annulus C_b not intersecting Z if and only if b is a good germ of segment of Y (in other words Z does not accumulate

[17] We notice that the annulus C' on which $\mathscr{F}$ is free of Liouville numbers is possibly not the same as the one on which $\mathrm{End}(\mathscr{F})$ is free of Liouville numbers. They may be disjoint, and we do not exclude the case where there are no loci on which both $\mathscr{F}$ and $\mathrm{End}(\mathscr{F})$ are free of Liouville numbers.

at b). In this case, the restriction $\mathcal{F}_{|C_b} = \mathscr{F}_{|C_b}$ has no meromorphic singularities and the notions developed in the previous section apply.

As a general principle, when speaking about the exponents and Liouville conditions, we say that $\mathcal{F}$ has a given property if $\mathscr{F}$ has that property. In particular, in the paper we will speak freely about the exponents of $\mathcal{F}$ (that are by definition those of $\mathscr{F}$) and the fact that $\mathcal{F}$ is free or strongly free of Liouville numbers (which means that $\mathscr{F}$ is).

A.3. Decomposition theorem of type Fuchs.

In this section we provide some statements about Liouville conditions that will be systematically used in the sequel.

The following theorem extends the major result of [CM97] to the case of pseudo-annuli.

Theorem A.3.1. *Assume that K is spherically complete, algebraically closed and that $|K| = \mathbb{R}_{\geqslant 0}$.*

Let $0 \leqslant r_1 < r_2 \leqslant +\infty$ and let $\mathscr{F}$ be a differential equation over the open pseudo-annulus $C := \{r_1 < |T| < r_2\}$, with log-affine radii along Γ_C. Assume that the exponent of $\mathscr{F}^{\mathrm{Robba}}$ has non-Liouville differences. Then $\mathscr{F}^{\mathrm{Robba}}$ is a successive extension of rank one differential modules.

More precisely, choose a multiset $\mathfrak{e}'$ of $(K^\circ/\mathbb{Z})^r$ lifting $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}})$. Choose a subset $\mathfrak{e}''$ of K° that contains exactly one representative of each element of $\mathfrak{e}'$. Then, we have

$$\mathscr{F}^{\mathrm{Robba}} \;=\; \bigoplus_{e\in\mathfrak{e}''} E(e)\;, \tag{A.25}$$

where $E(e)$ is a successive extension of the rank one module $\mathscr{N}(e)$ (cf. (A.7)) with dimension equal to the multiplicity of $e \mod \mathbb{Z}$ in $\mathfrak{e}'$.

Proof. If the residue characteristic of K is 0, then the claim follows from Remark A.1.29.

If the residue characteristic of K is positive, then the claim is proved in [CM97] when $0 < r_1 < r_2 < +\infty$. The general case follows easily. □

Corollary A.3.2. *Let C be an open pseudo-annulus and let $\mathscr{F}$ be a differential equation with log-affine radii along Γ_C that is free of Liouville numbers. Then C has finite-dimensional de Rham cohomology and we have*

$$\chi_{\mathrm{dR}}(C,\mathscr{F}) \;=\; 0\;. \tag{A.26}$$

For each germ of segment b in the skeleton Γ_C, $\mathscr{F}$ and $\mathscr{F}^{\mathrm{Robba}}$ are Fredholm at b and one has

$$\chi_b^{\mathrm{abs}}(\mathscr{F}) \;=\; \mathrm{Irr}_b(\mathscr{F}) \quad\textit{and}\quad \chi_b^{\mathrm{abs}}(\mathscr{F}^{\mathrm{Robba}}) \;=\; 0\;. \tag{A.27}$$

In particular, $\mathscr{F}$ satisfies $(\mathrm{Fin})_b$.

Moreover, if $C' \subseteq C$ is an inclusion of open pseudo-annuli such that $\Gamma_{C'} \subseteq \Gamma_C$, then, for $i = 0, 1$, the natural restriction

$$\mathrm{H}^i_{\mathrm{dR}}(C,\mathscr{F}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(C',\mathscr{F}_{|C'}) \tag{A.28}$$

is an isomorphism.

Proof. We can assume that K is algebraically closed, spherically complete and that $|K| = \mathbb{R}_{\geqslant 0}$. This follows from Theorem 1.4.7 for the part about cohomology and from the definitions for the part about the indexes. We may also assume that $\mathscr{F} = \mathscr{F}^{\mathrm{Robba}}$ by Propositions 2.3.5 and 2.3.10.

By Theorem A.3.1, $\mathscr{F}$ is then extension of rank one modules of type $\mathscr{N}(e)$. We may assume that $\mathscr{F}$ has rank one, by the five lemma for the part about cohomology and by Lemma 2.2.8 for the part about the indexes. In this case, the results follow from Lemma A.1.12.

Let us now prove the final statement. The map (A.28) is clearly injective for $i = 0$ and surjective for $i=1$ by Lemma 1.4.8. Since the indexes are zero on C and C', the map (A.28) is necessarily an isomorphism. $\square$

The following result is now a direct consequence of Corollary 2.3.19.

Corollary A.3.3. *Let C be an open pseudo-annulus and let b_0 and b_1 be the germs of segment at its open boundary. Let $\mathscr{F}$ be a differential equation over C. Assume that $\mathscr{F}$ is free of Liouville numbers at b_0 and b_1.*

Then, the following assertions are equivalent:

a) for each $i \geqslant 0$, $\mathrm{H}^i_{\mathrm{dR}}(C,\mathscr{F})$ is finite dimensional;

b) the total height of $\mathscr{F}$ is log-affine along b_0 and b_1.

Moreover, when these properties hold, we have

$$\chi_{\mathrm{dR}}(C,\mathscr{F}) \;=\; \mathrm{Irr}_{b_0}(\mathscr{F})+\mathrm{Irr}_{b_1}(\mathscr{F})\,. \tag{A.29}$$

$\square$

A.4. Some useful result.

In this section we state some useful results that are frequently used in the paper.

Lemma A.4.1. *Let D be an open pseudo-disk. Denote by b its germ of segment at infinity. Let $\mathscr{F}$ be a differential equation over D of rank r. Assume that, for every $i\in\{1,\dots,r\}$, the i-th radius is log-affine on b and we have $\partial_b\mathcal{R}_i(-,\mathscr{F})=0$. Then, $\mathscr{F}$ is free of Liouville numbers along b.*

Proof. Let C be an open pseudo-annulus whose skeleton represents b and on which all the radii of $\mathscr{F}$ are constant. In particular, the radii $\mathcal{R}_i(-,\mathscr{F})$ (before localization to C) are either spectral non-solvable on b or over-solvable on b. We deduce that the Robba part $(\mathscr{F}_{|C})^{\mathrm{Robba}}$ is the restriction to C of the maximal submodule of $\mathscr{F}$ that is trivial over D. Hence $(\mathscr{F}_{|C})^{\mathrm{Robba}}$ is the trivial equation and $\mathscr{F}$ is free of Liouville numbers. $\square$

We have the following interesting consequence.

Corollary A.4.2. *Let $\mathscr{F}$ be a differential equation on a quasi-smooth K-analytic curve X. Let Γ be a subgraph of X such that the radii of $\mathscr{F}$ are locally constant on $X\setminus\Gamma$. Then, the following assertions are equivalent:*

i) $\mathscr{F}$ is free of Liouville numbers along all good germs of segments in X;

ii) $\mathscr{F}$ is free of Liouville numbers along all good germs of segments inside Γ. $\square$

Lemma A.4.3. *Let C be an open pseudo-annulus and let $\mathscr{F}$ be a differential equation on C. Assume that $\mathscr{F}$ (resp. $\mathrm{End}(\mathscr{F})$) is free of Liouville numbers along Γ_C. Let $\mathscr{F}'$ be a sub-quotient of $\mathscr{F}$. Then, $\mathscr{F}'$ (resp. $\mathrm{End}(\mathscr{F}')$) is free of Liouville numbers along Γ_C.*

Proof. We may assume that K has positive residue characteristic. Moreover, it is clear that if $\mathscr{F}'$ is a sub-quotient of $\mathscr{F}$ then $\mathrm{End}(\mathscr{F}')=(\mathscr{F}')^*\otimes\mathscr{F}'$ is a sub-quotient of $\mathrm{End}(\mathscr{F})=\mathscr{F}^*\otimes\mathscr{F}$, so it is enough to prove the result for $\mathscr{F}$.

Along Γ_C, all the radii are spectral, hence the family of radii of any sub-quotient belongs to the family of radii of the original module (see [Ked10, Theorem 10.6.2]). This shows that the radii of $\mathscr{F}'$ are all log-affine along Γ_C. The claim now follows from Lemma A.4.4 below. $\square$

Lemma A.4.4 ([CM97, Proposition 5.4-3, and Théorèmes 5.4-5, 5.4-6])**.** *Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be an exact sequence of differential equation over an open pseudo-annulus C. If $\mathscr{F}_2$ has* log*-affine radii along Γ_C, then so have $\mathscr{F}_1$ and $\mathscr{F}_3$. In this case, we have an exact sequence*

$$0 \to \mathscr{F}_1^{\mathrm{Robba}} \to \mathscr{F}_2^{\mathrm{Robba}} \to \mathscr{F}_3^{\mathrm{Robba}} \to 0 \tag{A.30}$$

and the exponent of $\mathscr{F}_2$ is the union of those of $\mathscr{F}_1$ and $\mathscr{F}_3$.

In particular, if K has positive residue characteristic, then:

i) The exponent of $\mathscr{F}_2^{\mathrm{Robba}}$ is non-Liouville if, and only if, so are the exponents of $\mathscr{F}_1^{\mathrm{Robba}}$ and $\mathscr{F}_3^{\mathrm{Robba}}$.

ii) If the exponent of $\mathscr{F}_2^{\mathrm{Robba}}$ has non-Liouville differences, then so have the exponents of $\mathscr{F}_1^{\mathrm{Robba}}$ and $\mathscr{F}_3^{\mathrm{Robba}}$.

iii) If each element of the exponent of $\mathscr{F}_1^{\mathrm{Robba}}$ belongs to the exponent of $\mathscr{F}_3^{\mathrm{Robba}}$ or the other way round and if the exponents of $\mathscr{F}_1^{\mathrm{Robba}}$ and $\mathscr{F}_3^{\mathrm{Robba}}$ both have non-Liouville differences, then so has $\mathscr{F}_2^{\mathrm{Robba}}$. □

A.5. A characterization of the exponents.

Here, we recall a well-known cohomological characterization of the exponents that is often useful.

Let $C := \{r_1 < |T| < r_2\}$, with $0 \leqslant r_1 < r_2 \leqslant +\infty$, and let $\mathscr{N}(e)$ be as in (A.7). For $k \geqslant 0$, we consider the differential module

$$\mathscr{L}_k \;:=\; \bigoplus_{i=0}^{k} \mathscr{O}(C) \cdot \log(T)^i \;, \tag{A.31}$$

where $\log(T)$ is a formal symbol and the connection is given by the action of d/dT on $\log(T)^i$ as one expects. The union

$$\mathscr{L} \;:=\; \bigcup_k \mathscr{L}_k \tag{A.32}$$

is a differential ring (algebraically isomorphic to the ring of polynomials in the indeterminate $\log(T)$ with coefficients in $\mathscr{O}(C)$).

The following characterization follows from Theorem A.3.1 and Lemma A.1.12.

Lemma A.5.1. *Let $C := \{r_1 < |T| < r_2\}$, with $0 \leqslant r_1 < r_2 \leqslant +\infty$. Let $\mathscr{F}$ be a differential equation with* log*-affine radii along Γ_C such that the exponent $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}})$ has non-Liouville differences. Then, an element $e \in K$ belongs to the multiset $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}})$ if, and only if, one has*

$$\mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e)) \;\neq\; 0\,. \tag{A.33}$$

In this case the multiplicity of e in the multiset $\mathfrak{e}(\mathscr{F}^{\mathrm{Robba}})$ equals the dimension

$$\dim \mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e) \otimes \mathscr{L}) \;=\; \lim_{k\to\infty} \dim \mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e) \otimes \mathscr{L}_k)\,. \tag{A.34}$$

The space $\mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e) \otimes \mathscr{L})$ is also the space of solutions of $\mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e)$ with values in the differential ring $\mathscr{L}$.

Proof. By Theorem 1.4.7, we can assume that K is algebraically closed, spherically complete, and $|K| = \mathbb{R}_{\geqslant 0}$. The first part of the claim, concerning (A.33), then follows from Theorem A.3.1.

We now prove the second part of the claim. Replacing $\mathscr{F}^{\mathrm{Robba}}$ with $\mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e)$ we can assume that $e = 0$. In this case the block $E(0)$ appearing in Theorem A.3.1 is represented in a basis by the equation $Y' = GY$, where G is a standard nilpotent matrix in the Jordan form. The solutions of this equations are linear combinations of $\{1, \log(T), \log(T)^2, \dots\}$. So $E(0)$ is trivialized by the differential ring $\mathscr{L}$. In other words $E(0) \otimes \mathscr{L}$ is a trivial differential module over $\mathscr{L}$. The limit expression (A.34) follows from the fact that $\mathrm{H}^0_{\mathrm{dR}}$ commutes with inductive limits. □

Remark A.5.2. *Sometimes the dimension of $\mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e) \otimes \mathscr{L}_k)$ can be computed with an index formula, and it happens that it is necessary to check the Liouville condition on $\mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e) \otimes \mathscr{L}_k$. For this, we observe that $\mathscr{L}_k$ is a successive extension of the trivial rank one equation $y' = 0$, hence $\mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e) \otimes \mathscr{L}_k$ is a successive extension of $\mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e)$. For this reason, $\mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e) \otimes \mathscr{L}_k$ is free of Liouville numbers along Γ_C if, and only if, so is $\mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e)$ (cf. item (iii) of Lemma A.4.4). Notice moreover that, by Lemma A.5.1, the exponents of $\mathscr{F}^{\mathrm{Robba}} \otimes \mathscr{N}(-e)$ are those of $\mathscr{F}^{\mathrm{Robba}}$ to which we subtract e component by component.*

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Jérôme Poineau jerome.poineau@unicaen.fr
Laboratoire de MathÃ©matiques Nicolas Oresme, Université de Caen Normandie, BP 5186, F - 14032 Caen Cedex

Andrea Pulita andrea.pulita@univ-grenoble-alpes.fr
Université Grenoble Alpes, Institut Fourier, CS 40700, 38058 Grenoble cedex 9