An introduction to the Berkovich line over $\mathbf{Z}$

Jérôme Poineau
IRMAR
Université de Rennes 1, Campus de Beaulieu
35042 Rennes Cedex, France
courriel : jerome.poineau@univ-rennes1.fr

In this text, we would like to provide an introduction to the analytic line over $\mathbf{Z}$ in the sense of V. G. Berkovich. This space has been defined in [1], §1.5 as well as the structure sheaf over it but, as far as we know, a comprehensive study remained to be carried out. To begin with, we recall the definition of the space and then give some of its properties, which are similar to those of analytic spaces over fields (usual complex analytic varieties in the case of $\mathbf{C}$ and Berkovich spaces for non-Archimedean fields). The last part of the text is devoted to applications. From the general theory, we derive properties of convergent arithmetic power series in the sense of D. Harbater (see [3]).

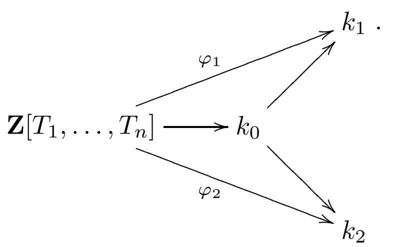
For the sake of simplicity, we have chosen to restrict our presentation to the analytic line over $\mathbf{Z}$. However, the constructions and the results remain valid over the ring of integers of a number field.

1 The affine analytic space over $\mathbf{Z}$

Let n be a non-negative integer. As a set, the scheme $\mathbf{A}^n_{\mathbf{Z}}$ is composed of the prime ideals of the ring $\mathbf{Z}[T_1, \ldots, T_n]$. Let us recall that this set is in bijection with the set of equivalence classes of morphisms

$$\mathbf{Z}[T_1, \ldots, T_n] \to k,$$

where k is a field. Two morphisms φ_1 and φ_2 are said to be equivalent if there exists a commuative diagram



If x is a point of the scheme $\mathbf{A}^n_{\mathbf{Z}}$, there exists a field, called the residue field, and a morphism $\mathbf{Z}[T_1, \ldots, T_n] \to k(x)$ that factorises all the morphisms represented by x.

In order to do analytic geometry, we need to have notions of norms and convergence. Therefore, we are going to modify slighty the previous definition and consider fields k that are endowed with an absolute value and complete. As regards the equivalence relations, we require the morphisms $k_0 \to k_1$ and $k_0 \to k_2$ to be isometric. The space we obtained this way is denoted by $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ (see [1], definition 1.5.1).

As in the case of schemes, if x is a point of $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$, the exists a complete valued field $\mathscr{H}(x)$ and a morphism

$$\mathbf{Z}[T_1, \ldots, T_n] \to \mathscr{H}(x)$$

that factorises all the morphisms represented by x.

This time, the set of equivalence classes is in bijection with the set of multiplicative semi-norms on $\mathbf{Z}[T_1, \ldots, T_n]$, *i.e.* the set of maps $|.| : \mathbf{Z}[T_1, \ldots, T_n] \to \mathbf{R}_+$ which satisfy the following properties:

i) $|0| = 0$ and $|1| = 1$;

ii) $\forall f, g \in \mathbf{Z}[T_1, \ldots, T_n]$, $|f + g| \leq |f| + |g|$;

iii) $\forall f, g \in \mathbf{Z}[T_1, \ldots, T_n]$, $|fg| = |f||g|$.

This bijection can be made explicit. To a morphism φ from $\mathbf{Z}[T_1, \ldots, T_n]$ to a complete valued field $(k, |.|)$, we associate the following multiplicative semi-norm:

$$\mathbf{Z}[T_1, \ldots, T_n] \xrightarrow{\varphi} k \xrightarrow{|.|} \mathbf{R}_+.$$

Now, let $|.|_x$ be a multiplicative semi-norm on $\mathbf{Z}[T_1, \ldots, T_n]$. The set $\mathfrak{p}_x$ where it vanishes is a prime ideal of $\mathbf{Z}[T_1, \ldots, T_n]$. The quotient is a domain on which the semi-norm $|.|_x$ induces an absolute value. We now define the field $\mathscr{H}(x)$ as the completion of the fraction field of this quotient. By construction, we have a morphism

$$\mathbf{Z}[T_1, \ldots, T_n] \to \mathscr{H}(x).$$

The topology on $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ will be the coarsest so that the evaluation maps, *i.e.* of the form

$$\begin{array}{ccc} \mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}} & \to & \mathbf{R}_+ \\ |.|_x & \mapsto & |P|_x \end{array},$$

with $P \in \mathbf{Z}[T_1, \ldots, T_n]$, are continuous.

Eventually, we endow the topological space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ with a sheaf of rings $\mathscr{O}$ (see [1], definition 1.5.3): a section over an open set U is an map

$$U \to \bigsqcup_{x \in U} \mathscr{H}(x)$$

which is locally a uniform limit of rational fractions with no poles.

2 The analytic line over Z

We will now be interested in the analytic line over $\mathbf{Z}$. Beforehand, it is useful to understand the space $\mathbf{A}_{\mathbf{Z}}^{0,\mathrm{an}}$, which will be denoted by $\mathscr{M}(\mathbf{Z})$. Thanks to Ostrowski's theorem, we have a very explicit description of the set of its points:

i) the trivial absolute value $|.|_0$;

ii) the Archimedean absolute values $|.|_{\infty,\varepsilon} = |.|_{\infty}^{\varepsilon}$, with $\varepsilon \in]0,1]$;

iii) for any prime number p, the p-adic absolute values $|.|_{p,\varepsilon} = p^{-\varepsilon v_p(\cdot)}$, with $\varepsilon \in]0,+\infty[$;

iv) for any prime number p, the semi-norm $|.|_{p,\infty} = |.|_p^{+\infty}$ induced by the trivial absolute value on $\mathbf{F}_p$.

Let us now describe the topology. Each of the p-adic branches is homeomorphic to the segment $[0,+\infty]$ and the Archimdean branch is homeomorphic to the segment $[0,1]$ (by the maps described above). We now have to describe the neighbourhoods of the point $|.|_0$: these are the subsets of $\mathscr{M}(\mathbf{Z})$ which contain entirely all the branches except a finite number of them and which contain a neighbourhood of $|.|_0$ in those that remain. In particular, the space $\mathscr{M}(\mathbf{Z})$ is compact.

We can also give an explicit description of the sections of the structure sheaf on the open sets of $\mathscr{M}(\mathbf{Z})$ (see figure 2).

Let us now go on and study the space $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$. Since every multiplicative semi-norm on $\mathbf{Z}[T]$ induces one on $\mathbf{Z}$, we have a projection morphism

$$\pi : \mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}} \to \mathscr{M}(\mathbf{Z}).$$

This way, we get a first topological description on the analytic line over Z: the fiber $\pi^{-1}(x)$ over a point x in $\mathscr{M}(\mathbf{Z})$ is isomorphic to the Berkovich line over the field $\mathscr{H}(x)$. If $\mathscr{H}(x) = \mathbf{R}$, this line is isomorphic to the quotient of the space $\mathbf{C}$ by complex conjugation. We will not try to give a more precise description but will instead point out some properties.

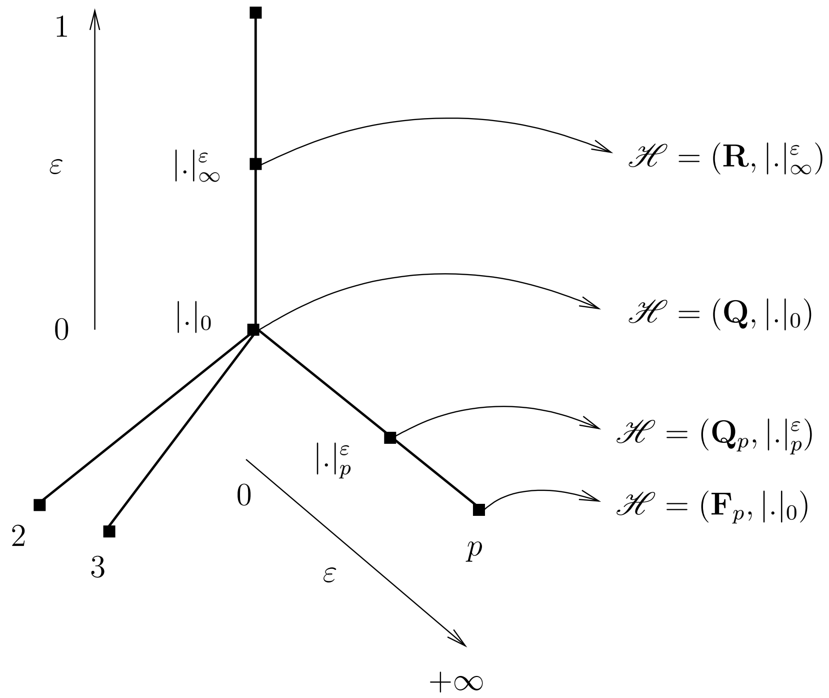


Figure 1: The space $\mathscr{M}(\mathbf{Z})$.

Theorem 2.1. *The analytic line* $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$ *is a Hausdorff, locally compact, arcwise connected, locally arcwise connected topological space of topological dimension* 3.

The projection morphism $\pi : \mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}} \to \mathscr{M}(\mathbf{Z})$ *is continuous and open.*

Let us now describe a few rings of sections over open sets of the analytic line $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$. Let Ω be an open set of $\mathscr{M}(\mathbf{Z})$ and r be a positive real number. We denote by $\|.\|_\Omega$ the uniform norm on Ω. We define an open set $\Omega(r)$ of $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$ by

$$\Omega(r) = \pi^{-1}(\Omega) \cap \{x \in \mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}} \,|\, |T(x)| < r\}.$$

The sections of the structure sheaf $\mathscr{O}_{\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}}$ over the open set $\Omega(r)$ can easily be described using the sections of $\mathscr{O}_{\mathscr{M}(\mathbf{Z})}$ over Ω: these are the series

$$\sum_{i\geq 0} a_i T^i \in \mathscr{O}(\Omega)[\![T]\!]$$

such that the radius of convergence of the series

$$\sum_{i\geq 0} \|a_i\|_\Omega \, T^i$$

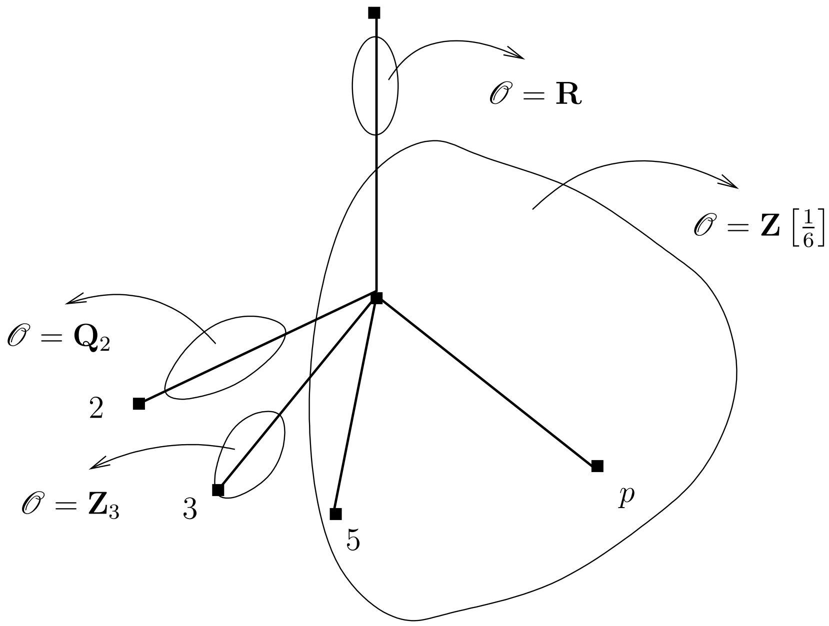


Figure 2: The structure sheaf of $\mathscr{M}(\mathbf{Z})$.

is at least r.

Let us define some notations. Let f be an element of $\mathbf{Q}[\![T]\!]$. We will denote by $R_\infty(f)$ the radius of convergence of the series f considered as an element of $\mathbf{C}[\![T]\!]$, where $\mathbf{C}$ is endowed with the usual absolute value $|.|_\infty$. For every prime number p, we will denote by $R_p(f)$ the radius of convergence of the series f considered as an element of $\mathbf{C}_p[\![T]\!]$, where $\mathbf{C}_p$ is endowed with the normalized p-adic absolute value $|.|_p$ ($|p|_p = \frac{1}{p}$).

Let t be a non-negative integer, $p_1, \ldots, p_t$ be prime numbers and $\varepsilon_1, \ldots, \varepsilon_t$ be positive real numbers. We assume that the open set Ω of $\mathscr{M}(\mathbf{Z})$ contains the interval $[|.|_0, |.|_{p_1,\varepsilon_1}[$ in the p_1-adic branch, …, the interval $[|.|_0, |.|_{p_t,\varepsilon_t}[$ in the p_t-adic branch and all the other branches entirely. Then the elements of $\mathscr{O}(\Omega(r))$ are exactly the series

$$f \in \mathbf{Z}\left[\frac{1}{p_1 \cdots p_t}\right][\![T]\!]$$

which satisfy the following conditions:

$$R_\infty(f) \geq r \text{ and } \forall i \in [\![1, t]\!],\ R_{p_i}(f) \geq r^{1/\varepsilon_i}.$$

3 Applications

The analytic space $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$ satisfies numerous properties similar to those of complex analytic spaces.

Theorem 3.1. *For every point x of $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$, the local ring $\mathscr{O}_{\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}},x}$ is Henselian, Noetherian and regular.*

The structure sheaf $\mathscr{O}$ is coherent.

Let us describe the local ring A at the point 0 of the fiber of π over the trivial absolute value $|.|_0$. It is the set of the elements f of $\mathbf{Q}[\![T]\!]$ that satisfy the following properties:

i) there exists $N \in \mathbf{N}^*$ such that $f \in \mathbf{Z}[1/N][\![T]\!]$;

ii) we have $R_\infty(f) > 0$;

iii) for every prime number p, we have $R_p(f) > 0$.

With a little work, one can deduce from the fact that this ring is Henselian a classical theorem due to Eisenstein: every element of $\mathbf{Q}[\![T]\!]$ which is integral over $\mathbf{Q}[T]$ belongs to A.

It is also possible to define a notion of Stein space for the subsets of $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$. Let P be a subset of $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$. We denote by $j : P \hookrightarrow \mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$ the inclusion morphism. We endow the space P with the sheaf of rings $j^{-1}\mathscr{O}$. If P is an open subset of $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$, it is only the restriction of the structure sheaf. If P is a closed subset of $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$, we are considering the sheaf of overconvergent functions.

Definition 3.2. We say that a subset P of $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}}$ is **Stein** if it satisfies theorem A (*for every coherent sheaf $\mathscr{F}$ over P and every point x in P, the stalk $\mathscr{F}_x$ is generated, as an $\mathscr{O}_x$-module, by the image of the set of global sections $\mathscr{F}(P)$*) and theorem B (*for every coherent sheaf $\mathscr{F}$ over P and every positive integer q, we have $H^q(P, \mathscr{F}) = 0$*).

Definition 3.3. We say that a subset of $\mathscr{M}(\mathbf{Z})$ is **semi-analytic** if it is locally defined by a boolean combination of inequalities between norms of functions.

Theorem 3.4. *Any open or closed disk or annulus over a semi-analytic subset of $\mathscr{M}(\mathbf{Z})$ is a Stein space.*

Doing as in complex analytic geometry, we deduce some applications. We denote by D the open unit disk in $\mathbf{C}$.

Theorem 3.5. *Let E and F be two non-intersecting, closed and discrete subsets of D that do not contain the point 0. Let $(n_a)_{a\in E}$ be a family of non-negative integers and $(P_b)_{b\in F}$ be a family of polynomials with no constant term. We assume that*

i) for every $a \in E$, $\bar{a} \in E$ and $n_{\bar{a}} = n_a$;

ii) for every $b \in F$, $\bar{b} \in F$ and $P_{\bar{b}} = \overline{P_b}$.

Then there exist $g, h \in \mathbf{Z}[\![T]\!] \cap \mathscr{O}(D)$ which satisfy the following properties :

a) the function $f = g/h$ is meromorphic on D and holomorphic on $D \setminus F$;

b) for every $a \in E$, the functionf vanishes in a with order at least n_a;

c) for every $b \in E$, we have $f(z) - P_b\left(\frac{1}{z-b}\right) \in \mathscr{O}_b$;

d) we have $f \in \mathbf{Z}[\![T]\!] \cap \mathscr{O}_0$.

The proof of this result makes use of cohomological methods. When the subset E is empty, it relies on the short exact sequence $0 \to \mathscr{O} \to \mathscr{M} \to \mathscr{M}/\mathscr{O} \to 0$ and on the fact that the open disk of center 0 and radius 1 in $\mathbf{A}^{1,\mathrm{an}}_{\mathbf{Z}}$ is Stein. When the subset E is non-empty, we replace the sheaf $\mathscr{O}$ by an appropriate Cartier divisor.

Let $\mathcal{P}$ be a finite set of prime number. We denote their product by $N \in \mathbf{N}^*$. It is also possible to fix, for any prime number $p \in \mathcal{P}$, the principal parts of the function f considered as a meromorphic function on the open unit disk in $\mathbf{C}_p$. Then the coefficients of g, h and the Taylor development of f at 0 will lie in the ring $\mathbf{Z}[1/N]$.

Finally, we can adapt a theorem of J. Frisch (*cf.* [2]) concerning the Noetherianity of rings of overconvergent functions. This theorem was originally proved for the semi-analytic subsets of real and complex analytic spaces. We replace it by the notion of morceable subset, which is very close.

Theorem 3.6. *Let L be a compact, morceable and Stein subset of $\mathbf{A}^{1,\mathrm{an}}_{\mathbf{Z}}$.Then the ring $\mathscr{O}(L)$ of functions that are analytic in the neighbourhood of L is Noetherian.*

If we apply this theorem to compact disks over compact and connected neighbourhoods of $|.|_0$ in $\mathscr{M}(\mathbf{Z})$, we prove that for any non-negative integer t, any prime numbers $p_1, \ldots, p_t$ and any positive real numbers $\varepsilon_1, \ldots, \varepsilon_t, r$, the ring composed of the series of the form

$$f \in \mathbf{Z}\left[\frac{1}{p_1 \cdots p_t}\right][\![T]\!]$$

which satisfy the conditions

$$R_\infty(f) > r \text{ and } \forall i \in [\![1, t]\!],\ R_{p_i}(f) > r^{1/\varepsilon_i}$$

is Noetherian.

When $t = 0$, we find a result of D. Harbater (see [3], theorem 1.8).

References

[1] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[2] Jacques Frisch. Points de platitude d'un morphisme d'espaces analytiques complexes. *Invent. Math.*, 4:118–138, 1967.

[3] David Harbater. Convergent arithmetic power series. *Amer. J. Math.*, 106(4):801–846, 1984.