The convergence Newton polygon of a p-adic differential equation V: global index theorems

Jérôme Poineau and Andrea Pulita

Contents

2000 Mathematics Subject Classification Primary 12h25; Secondary 14G22

Keywords: p-adic differential equations, Berkovich spaces, de Rham cohomology, index, irregularity, radius of convergence, Newton polygon, Grothendieck-Ogg-Shafarevich formula, super-harmonicity, Banach spaces

Here and for the rest of the text, we fix an ultrametric complete valued field $(K, |\cdot|)$ of characteristic 0. We denote by $\widetilde{K}$ its residue field and by p be the characteristic exponent of the latter (either 1 or a prime number). We fix an algebraic closure K^{alg} of K. The absolute value $|\cdot|$ on K extends uniquely to it and we denote it identically. We denote by $(\widehat{K^{\mathrm{alg}}}, |\cdot|)$ its completion.

We also set $\omega := \liminf_n |n!|^{1/n}$ (the radius of convergence of the exponential series). One has

$$\omega = \begin{cases} 1 & \text{if the valuation of } K \text{ is trivial on } \mathbb{Q}, \\ |p|^{\frac{1}{p-1}} & \text{if the valuation of } K \text{ is } p\text{-adic on } \mathbb{Q}. \end{cases} \tag{0.1}$$

1. Definitions and notations

In the whole paper, we will use the definitions and notation from [PP13] and we refer the reader to this manuscript (and especially to its first section). We recall here the most important.

1.1. Radii of convergence

Definition 1.1.1 (Virtual open disc). *A non-empty connected K-analytic space is called a virtual open disk if it becomes isomorphic to a disjoint union of open discs over $\widehat{K^{\mathrm{alg}}}$.*

Following [Duc, 5.1.8], we may now define the analytic skeleton of an analytic curve.

Definition 1.1.2 (Analytic skeleton). *We call* analytic skeleton *of an analytic curve X the set of points that have no neighborhoods isomorphic to a virtual open disk. We usually denote it by Γ_X.*

Definition 1.1.3 (Open pseudo-annulus). *Assume that K is algebraically closed. We say that a connected quasi-smooth K-analytic curve C is an* open pseudo-annulus *if*

i) it has no boundary;

ii) it contains no points of positive genus;

iii) its analytic skeleton Γ_C is an open segment.

We call Γ_C the skeleton of C.

If K is arbitrary, we say that a connected quasi-smooth K-analytic curve C is an open pseudo-annulus *if $C\hat{\otimes}_K\widehat{K^{\mathrm{alg}}}$ is a disjoint union of open pseudo-annuli and if Gal(K^{alg}/K) preserves the orientation of their skeletons. In this case, one can check that C has no boundary, contains no points of positive genus and that its analytic skeleton Γ_C is an open segment. We call Γ_C the skeleton of C.*

Let X be a quasi-smooth K-analytic curve.

Definition 1.1.4 (Pseudo-triangulation). *A pseudo-triangulation of X is a locally finite subset $S \subset X$, formed by points of type 2 or 3, such that every connected component of $X - S$ is a virtual open disk or an open pseudo-annulus.*

The skeleton Γ_S of a pseudo-triangulation S is the union of S with the skeletons of the connected components of $X - S$ that are open pseudo-annuli.

For the rest of the section, we fix a pseudo-triangulation S of X.

Let $(\mathscr{F}, \nabla)$ be a module with connection over X. The pseudo-triangulation S on X is the the datum we will use to normalize the radii of convergence of $(\mathscr{F}, \nabla)$ at the points of X. Let us recall quickly how to do it.

Let $x \in X$ and let r be the rank of $\mathscr{F}$ at x. Let L be a complete algebraically closed valued field containing $\mathscr{H}(x)$. In this case, there exists an L-rational point x' of X_L over X. One can prove that the pseudo-triangulation S of X induces a pseudo-triangulation S_L of X_L. Let D be the connected component of $X_L - S_L$ containing x'. It is an open disk and we identify it to some $D(0, R)^-$ by some isomorphism sending x' to 0.

For $i \in \{1, \dots, r\}$, denote by $\mathcal{R}'_i$ the supremum of the radii of the closed disks centered at 0 on which the differential equation induced by $(\mathscr{F}, \nabla)$ admits at least $r - i + 1$ linearly independent solutions. It is strictly positive.

Definition 1.1.5. *For $i \in \{1, \dots, r\}$, the i^{th} radius of convergence of $(\mathscr{F}, \nabla)$ at x is*

$$\mathcal{R}_{S,i}(x, (\mathscr{F}, \nabla)) := \frac{\mathcal{R}'_i}{R} \in \,]0, 1].$$

It is independent of the choices made.

The total height of the Newton polygon of $(\mathscr{F}, \nabla)$ at x is

$$H_{S,r}(x, (\mathscr{F}, \nabla)) = \prod_{i=1}^{r} \mathcal{R}_{S,i}(x, (\mathscr{F}, \nabla)).$$

Let $x \in X \setminus \Gamma_S$. Let y be a point of $X_{\widehat{K^{\mathrm{alg}}}}$ over x. Let E_y be the connected component of $X_{\widehat{K^{\mathrm{alg}}}} \setminus \Gamma_{S_{\widehat{K^{\mathrm{alg}}}}}$ containing y. It is an open disc and we identify it to some $D(0, \rho)^-$. Denoting by $r(y)$ the radius of the point y in $D(0, \rho)^-$ (see REF), we set

$$\rho_S(x) := \frac{r(y)}{\rho} \in [0, 1). \tag{1.1}$$

It is independent of the choices made.

For $x \in \Gamma_S$, we set

$$\rho(x) := 1. \tag{1.2}$$

Definition 1.1.6. *Let $x \in X$. For $i \in \{1, \dots, r\}$, the i^{th} radius of convergence of $(\mathscr{F}, \nabla)$ at x is said to be* spectral *(resp.* solvable*) if we have $\mathcal{R}_{S,i}(x, (\mathscr{F}, \nabla)) \leqslant \rho_S(x)$ (resp. $\mathcal{R}_i(x, (\mathscr{F}, \nabla)) = \rho_S(x)$).*

The radii of convergence satisfy nice properties. The main result of [**?**] may be stated as follows.

Theorem 1.1.7. *Let $i \in \{1, \dots, r\}$. The map*

$$\mathcal{R}_{S,i}(-, (\mathscr{F}, \nabla)) \colon X \to \,]0, 1] \tag{1.3}$$

is continuous on X and piecewise log-affine with rational slopes on each segment inside X.

Moreover, there exists a locally finite subgraph of X outside of which the map $\mathcal{R}_{S,i}(-,(\mathscr{F},\nabla))$ is locally constant. □

We denote by $\Gamma_S((\mathscr{F},\nabla))$ the smallest subgraph of X containing Γ_S outside which the maps $\mathcal{R}_{S,i}(-,(\mathscr{F},\nabla))$, for $i \in \{1,\dots,r\}$, are all constant.

In the rest of the text, we will often write $\mathscr{F}$ instead of $(\mathscr{F},\nabla)$. This should lead to no confusion.

1.2. Stein spaces.

In this section, we recall some result about Stein curves from [**?**, Section 4.1].

Definition 1.2.1 (Cohomologically Stein, [**?**, Definition 4.5])**.** *We say that a K-analytic space X is* cohomologically Stein *if, for every coherent sheaf $\mathscr{F}$ on X and every $q \geqslant 1$, we have*

$$H^q(X,\mathscr{F}) = 0. \tag{1.4}$$

Theorem 1.2.2 ([**?**, Corollaries 4.6 and 4.8])**.** *Let X be a quasi-smooth K-analytic curve with no proper connected components. Then X is cohomologically Stein and every coherent sheaf $\mathscr{F}$ on X is generated by its global sections: for each $x \in X$, the stalk $\mathscr{F}_x$ is generated by $\mathscr{F}(X)$ as an $\mathscr{O}_{X,x}$-module.* □

We say that a coherent sheaf $\mathscr{F}$ on a K-analytic space X is of bounded rank if the family $\big(\mathrm{rank}_{\mathscr{H}(x)}(\mathscr{F}(x))\big)_{x\in X}$ is bounded.

Corollary 1.2.3 ([**?**, Corollary 4.9])**.** *Let X be a quasi-smooth K-analytic curve with no proper connected components. Let $\mathscr{F}$ be a coherent sheaf of bounded rank on X. Then the module of global sections $\mathscr{F}(X)$ is of finite type over $\mathscr{O}(X)$.*

In particular, there exist an integer q and a surjective morphism $\mathscr{O}^q \to \mathscr{F}$. □

Corollary 1.2.4 ([**?**, Corollary 4.11])**.** *Let X be a quasi-smooth K-analytic curve with no proper connected components. The functor $\mathscr{F} \mapsto \mathscr{F}(X)$ induces an equivalence between the category of coherent sheaves of bounded rank (resp. locally free sheaves of bounded rank) and the category of $\mathscr{O}(X)$-modules of finite type (resp. projective $\mathscr{O}(X)$-modules of finite type).* □

1.3. Tubes.

In this section, we give some definitions inspired by those in rigid cohomology. Let X be a quasi-smooth K-analytic curve, S be a pseudo-triangulation of X and $(\mathscr{F},\nabla)$ be a module with connection on X.

Definition 1.3.1 (Elementary tube)**.** *Let $x \in X$. A connected analytic domain V of X is said to be an* elementary tube centered at x *if $V - \{x\}$ is a disjoint union of virtual open disks.*

It is said to be an elementary tube adapted to $\mathscr{F}$ *if, moreover, the radii of $\mathscr{F}$ are constant on each of these virtual open disks.*

Remark 1.3.2. *If x is a point of type 1 or 4, there are no elementary tubes centered at x. Indeed there are no disks in X with boundary x.*

If x has type 3, there are three possibilities for an elementary tube V centered at x:

i) $V = \{x\}$;

ii) V is a virtual closed disk with boundary x;

iii) V is a connected component of X isomorphic to the analytification of a projective curve of *genus 0.*

If x has type 2, then V is either an affinoid domain or a connected component of X isomorphic to the analytification of a projective curve.

Definition 1.3.3 (Singular directions)**.** *Let $x \in X$ and let V be an elementary tube centered at x. There are a finite number of germs of segments $b_1, \ldots, b_n$ out of x that do not meet V. We call them* singular directions *at x with respect to V and write*

$$\mathrm{Sing}(x, V) \;:=\; \{b_1, \ldots, b_n\}\;. \tag{1.5}$$

If K is algebraically closed, we set

$$N^s_V(x) \;:=\; n \;=\; \mathrm{Card}(\mathrm{Sing}(x, V))\;. \tag{1.6}$$

If K is arbitrary, we denote by $x'_1, \ldots, x'_d$ the points of $X_{\widehat{K^{\mathrm{alg}}}}$ over x. For every $i \in \{1, \ldots, d\}$, the connected component V'_i of $V_{\widehat{K^{\mathrm{alg}}}}$ containing x'_i is an elementary tube centered at x'_i. We set

$$N^s_V(x) \;:=\; d\, N^s_{V'_1}(x'_1) = \cdots = d\, N^s_{V'_d}(x'_d) = \sum_{i=1}^{d} N^s_{V'_i}(x'_i)\;. \tag{1.7}$$

Definition 1.3.4 (Elementary neighborhoods)**.** *Let Y be an analytic domain of X. For every $y \in \partial Y$, denote by $b_{y,1}, \ldots, b_{y,t_y}$ the germs of segments out of x that belong to X but not to Y. We say that an open subset U of X is an* elementary neighborhood *of Y if it may be written in the form*

$$U \;=\; Y \cup \Big(\bigcup_{\substack{y\in\partial Y\\ 1\leqslant i\leqslant t_y}} C_{y,i}\Big)\;, \tag{1.8}$$

where $C_{y,i}$ is a non-empty virtual open annulus whose skeleton (suitably oriented) represents $b_{y,i}$ and all the $C_{y,i}$'s are disjoint and disjoint from Y.

The open subset U is said to be an elementary neighborhood adapted to *$\mathscr{F}$ if, moreover, for every $y \in \partial Y$ and every $i \in \{1, \ldots, t_y\}$, the radii of $\mathscr{F}_{|C_{y,i}}$ are log-affine on the skeleton of $C_{y,i}$ (where $C_{y,i}$ is endowed with the empty pseudo-triangulation).*[1]

Remark 1.3.5. *i) If S_Y is a pseudo-triangulation of Y, then it is also a pseudo-triangulation of all of its elementary neighborhoods. In particular, if V is an elementary tube centered at x, then $\{x\}$ is a pseudo-triangulation of every elementary neighborhood of V. In the sequel, this will always be the triangulation we choose for elementary neighborhoods.*

ii) The set of elementary neighborhoods of an analytic domain Y that are adapted to $\mathscr{F}$ is a basis of neighborhoods of Y.

We also recall the following result.

Proposition 1.3.6 ([PP13, Corollary 6.2.28])**.** *Let C be an open pseudo-annulus endowed with the empty pseudo-triangulation $S = \emptyset$. Let $\mathscr{F}$ be a differential equation over C with log-affine radii along Γ_C. Then, for all $x \in X$, the radii of $\mathscr{F}$ are locally constant on $C \setminus \Gamma_C$.* □

For later use, it is convenient to introduce a notion of overconvergent Euler characteristic. We now do so.

[1] See Section 2.1.2 for the case with meromorphic singularities.

Lemma 1.3.7. *Let V be an analytic domain of X that has a finite number of connected components. There exists an open subset U of X containing V such that $U - V$ is a disjoint union of pseudo-annuli C, such that the relative boundary of each C in X has exactly two points and exactly one of them lies in ∂V.*

Moreover, the open sets U as above form a basis of neighborhood of V in X and they all have the same compactly supported Euler characteristic. □

Definition 1.3.8. *With the notations of Lemma 1.3.7, we define the overconvergent compactly supported Euler characteristic of V by*

$$\chi_c(V^\dagger, X) \ := \ \chi_c(U)\ . \tag{1.9}$$

Usually, when X is clear from the context, we will simply write $\chi_c(V^\dagger)$ instead of $\chi_c(V^\dagger, X)$.

Recall that, when x is a point of $X - \partial X$ of type 2, there is a natural bijection between the set of germs of segments out of x and the set of closed points of $\mathcal{C}_x$ (see [Duc, Théorème 4.2.10]).

Lemma 1.3.9. *Assume that K is algebraically closed. Let $x \in \mathrm{Int}(X)$ be a point of type 2, let V be an elementary tube centered at x and let U be an elementary neighborhood of V. Let $p_1, \dots, p_{N^s_V(x)}$ be the closed points of $\mathcal{C}_x$ corresponding to the singular directions $b_1, \dots, b_{N^s_V(x)}$ at x. Denote by $\chi_c(\mathcal{C}_x - \{p_1, \dots, p_{N^s_V(x)}\})$ the compactly supported Euler characteristic of the curve $\mathcal{C}_x - \{p_1, \dots, p_{N^s_V(x)}\}$ in the sense of étale cohomology. We have*

$$\chi_c(V^\dagger) = \chi_c(U) = \chi_c(\mathcal{C}_x - \{p_1, \dots, p_{N^s_V(x)}\}) = 2 - 2g(x) - N^s_V(x)\ . \tag{1.10}$$

□

Recall that the graph $\Gamma_S(\mathscr{F})$ is locally finite thanks to Theorem 1.1.7.

Definition 1.3.10 (Canonical tube and canonical singular directions)**.** *Let $x \in X$ be a point of type 2 or 3. We denote by $V_S(x, \mathscr{F})$ the union of $\{x\}$ with all the virtual open disks with boundary x that do not meet $\Gamma_S(\mathscr{F})$.*

The set $V_S(x, \mathscr{F})$ is an elementary tube centered at x that is adapted to $\mathscr{F}$. We call it the canonical tube *at x.*

We define the set of canonical singular directions *at x to be (cf. Definition 1.3.3)*

$$\mathrm{Sing}_S(x, \mathscr{F}) \ := \ \mathrm{Sing}(x, V_S(x, \mathscr{F}))\ . \tag{1.11}$$

We set

$$N^s_S(x, \mathscr{F}) \ := \ N^s_{V_S(x,\mathscr{F})}(x)\ . \tag{1.12}$$

In the sequel, we will often suppress the subscript S when it is clear from the context.

Remark 1.3.11. *Assume that K is algebraically closed. Let $x \in X$ be a point of type 2 or 3. Then, $N^s_S(x, \mathscr{F})$ is equal to the number of branches out of x belonging to $\Gamma_S(\mathscr{F})$ if $x \in \Gamma_S(\mathscr{F})$, and to 1 otherwise.*

Definition 1.3.12 (Maximal tube and maximal singular directions)**.** *Let $x \in X$ be a point of type 2 or 3. We denote by $V_{\mathrm{m}}(x, \mathscr{F})$ the union of $\{x\}$ with all the connected components D of $X - \{x\}$ with boundary x such that*

i) D is a virtual open disk;

ii) the radii of convergence of $\mathscr{F}_{|D}$, where D is endowed with the empty pseudo-triangulation, are constant.

The set $V_{\mathrm{m}}(x,\mathscr{F})$ *is an elementary tube centered at* x*. We call it the* maximal tube *at* x*.*

We define the set of maximal singular directions *at* x *to be*

$$\mathrm{Sing}_m(x,\mathscr{F}) \;:=\; \mathrm{Sing}(x,V_{\mathrm{m}}(x,\mathscr{F}))\;. \tag{1.13}$$

Remark 1.3.13. *Let* S *be a pseudo-triangulation of* X*. There exists a pseudo-triangulation* S' *of* X*, contained in* S *such that* $V_{S'}(x,\mathscr{F})=V_{\mathrm{m}}(x,\mathscr{F})$*. Indeed, it is enough to remove from* S *all the points contained in* $V_{\mathrm{m}}(x,\mathscr{F})-\{x\}$*.*

It follows that

$$V_{\mathrm{m}}(x,\mathscr{F})=\bigcup_T V_T(x,\mathscr{F}), \tag{1.14}$$

where T *runs through all the triangulations of* X*, whence the name.*

The following corollary is often useful to remove the boundary.

Corollary 1.3.14. *Let* Z *be a locally finite subset of rigid points of* X*. Let* $\mathcal{F}$ *be a differential equation on* X *with meromorphic singularities on* Z *and assume that all its radii are spectral non-solvable on* ∂X*. For every* $x\in\partial X$*, choose an elementary tube* V_x *in* $X-Z$ *centered at* x *that is adapted to* $\mathcal{F}_{|X-Z}$ *(cf. Definition 1.3.1). Set*

$$Y \;:=\; X-\bigcup_{x\in\partial X} V_x\;. \tag{1.15}$$

Then, for $i=0,1$*, we have canonical isomorphisms*

$$\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(\mathrm{Int}(X)(*Z),\mathcal{F}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(Y(*Z),\mathcal{F})\;. \tag{1.16}$$

Proof. Let $x\in\partial X$ and let U_x be an elementary neighborhood of V_x in $X-Z$ that is adapted to $\mathcal{F}_{|X-Z}$ (cf. Definition 1.3.4). By Remark 1.5.3 and Corollary 1.7.2, for every $i\geqslant 0$, we have $\mathrm{H}^i_{\mathrm{dR}}(U_x(*Z),\mathcal{F}_{|U_x})=\mathrm{H}^i_{\mathrm{dR}}(U_x,\mathcal{F}_{|U_x})=0$. Up to shrinking the U_x's, we may assume that they are all disjoint. Set $U:=\bigcup_{x\in\partial X}U_x$. Then, for every $i\geqslant 0$, we have $\mathrm{H}^i_{\mathrm{dR}}(U(*Z),\mathcal{F}_{|U})=\mathrm{H}^i_{\mathrm{dR}}(U,\mathcal{F}_{|U})=0$.

The intersection $U\cap Y$ is a disjoint union of virtual open annuli on which the radii are all spectral non-solvable and log-affine, hence, by Proposition 1.7.1, we have $\mathrm{H}^i_{\mathrm{dR}}((U\cap Y)(*Z),\mathcal{F}_{|U\cap Y})=\mathrm{H}^i_{\mathrm{dR}}(U\cap Y,\mathcal{F}_{|U\cap Y})=0$ for all $i\geqslant 0$.

Since $\{Y,U\}$ is a covering of X, the isomorphism $\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F})\xrightarrow{\sim}\mathrm{H}^i_{\mathrm{dR}}(Y(*Z),\mathcal{F}_{|Y})$ now follows from the Mayer-Vietoris exact sequence (see Lemma 1.4.3).

Finally, $\mathrm{Int}(X)=X-\partial X$ is the disjoint union of Y and of a family of disjoint virtual open disks not meeting Z on which the radii are all constant. For every such disk D, the non-solvability assumption implies that $\mathrm{H}^0_{\mathrm{dR}}(D(*Z),\mathcal{F}_{|D})=\mathrm{H}^0_{\mathrm{dR}}(D,\mathcal{F}_{|D})=0$ and it follows from Corollary **??** that we also have $\mathrm{H}^1_{\mathrm{dR}}(D(*Z),\mathcal{F}_{|D})=\mathrm{H}^1_{\mathrm{dR}}(D,\mathcal{F}_{|D})=0$. The result now follows from the Mayer-Vietoris exact sequence as before. □

1.4. de Rham cohomology

1.4.1. Definition and classical results Let X be a quasi-smooth K-analytic curve and let $\mathscr{F}$ be a differential equation on X. Consider the complex of sheaves

$$\mathcal{E}(\mathscr{F})^\bullet\colon (\cdots\to 0\to\mathscr{F}\xrightarrow{\nabla}\Omega^1_X\otimes\mathscr{F}\to 0\to\cdots)\;, \tag{1.17}$$

where $\mathscr{F}$ is placed in degree 0 and $\Omega^1_X\otimes\mathscr{F}$ in degree 1. The cohomology of $\mathscr{F}$ (resp. the hypercohomology of $\mathcal{E}(\mathscr{F})^\bullet$) will be denoted by $H^i(X,\mathscr{F})$ (resp. $\mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^\bullet)$).

Remark 1.4.1. *In our situation,* X *is a* K*-analytic curve, hence has topological dimension 1. It* *follows that $H^i(X,\mathscr{F})=0$ for $i\geqslant 2$ and that $\mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^\bullet)=0$ for $i\geqslant 3$ (by a spectral sequence argument).*

Definition 1.4.2. *The de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ of $\mathscr{F}$ are the hypercohomology groups $\mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^\bullet)$ of the complex $\mathcal{E}(\mathscr{F})^\bullet$:*

$$\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \;:=\; \mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^\bullet)\;. \tag{1.18}$$

We say that $\mathscr{F}$ has finite index *if $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ is finite-dimensional for all degrees $i\in\mathbb{Z}$. In this case we denote by $\mathrm{h}^i_{\mathrm{dR}}(X,\mathscr{F})$ the dimension of the K-vector space $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ and set*

$$\begin{aligned}\chi_{\mathrm{dR}}(X,\mathscr{F}) \;&:=\; \sum_i (-1)^i\cdot \mathrm{h}^i_{\mathrm{dR}}(X,\mathscr{F}) &(1.19)\\ &=\; \mathrm{h}^0_{\mathrm{dR}}(X,\mathscr{F})-\mathrm{h}^1_{\mathrm{dR}}(X,\mathscr{F})+\mathrm{h}^2_{\mathrm{dR}}(X,\mathscr{F})\;. &(1.20)\end{aligned}$$

We call $\chi_{\mathrm{dR}}(X,\mathscr{F})$ the index of $\mathscr{F}$.

Lemma 1.4.3 (Mayer-Vietoris)**.** *Let U and V be two open subsets of X such that $X=U\cup V$. We have the Mayer-Vietoris long exact sequence*

$$\cdots\to \mathrm{H}^{i-1}_{\mathrm{dR}}(U\cap V,\mathscr{F})\to \mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})\to \mathrm{H}^i_{\mathrm{dR}}(U,\mathscr{F})\oplus\mathrm{H}^i_{\mathrm{dR}}(V,\mathscr{F})\to \mathrm{H}^i_{\mathrm{dR}}(U\cap V,\mathscr{F})\to \mathrm{H}^{i+1}_{\mathrm{dR}}(X,\mathscr{F})\to\cdots \tag{1.21}$$

In particular, if, for all $i\in\mathbb{Z}$, the spaces $\mathrm{H}^i_{\mathrm{dR}}(U,\mathscr{F})$, $\mathrm{H}^i_{\mathrm{dR}}(V,\mathscr{F})$ and $\mathrm{H}^i_{\mathrm{dR}}(U\cap V,\mathscr{F})$ are finite-dimensional, then, for all $i\in\mathbb{Z}$, the space $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ is finite-dimensional too. □

Lemma 1.4.4. *If X is cohomologically Stein, then we have*

$$\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F}) \;=\; \mathrm{Ker}(\nabla\colon \mathscr{F}(X)\to\Omega^1(X)\otimes_{\mathscr{O}(X)}\mathscr{F}(X)) \tag{1.22}$$

and

$$\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) \;=\; \mathrm{Coker}(\nabla\colon \mathscr{F}(X)\to\Omega^1(X)\otimes_{\mathscr{O}(X)}\mathscr{F}(X))\;, \tag{1.23}$$

and $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})=0$ for all $i\neq 0,1$. □

Using the snake lemma, we obtain the following application.

Lemma 1.4.5 (Additivity of index)**.** *Assume that X is cohomologically Stein. Let $0\to\mathscr{F}\to\mathscr{G}\to\mathscr{H}\to 0$ be a short exact sequence of modules with connections on X. If two among $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ have finite index, then so has the third. In this case, we have*

$$\chi(X,\mathscr{G}) \;=\; \chi(X,\mathscr{F})+\chi(X,\mathscr{H})\;. \tag{1.24}$$

□

1.4.2. Properties

We gather here a few properties that will be used later in the text.

Lemma 1.4.6 ([?, Lemma 1.4.6])**.** *If X has finitely many connected components, then $\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F})$ is finite-dimensional.* □

Theorem 1.4.7 ([?, Corollary 4.14])**.** *Let L be a complete valued extension of K. Assume that there exists $M\in\{K,L\}$ such that M is not trivially valued and $\mathrm{H}^1_{\mathrm{dR}}(X_M,\mathscr{F}_M)$ is finite-dimensional. Then, $\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F})$ and $\mathrm{H}^1_{\mathrm{dR}}(X_L,\mathscr{F}_L)$ are both finite-dimensional and we have natural isomorphisms*

$$\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F})\otimes_K L\xrightarrow{\sim}\mathrm{H}^0_{\mathrm{dR}}(X_L,\mathscr{F}_L)\quad\text{and}\quad \mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F})\otimes_K L\xrightarrow{\sim}\mathrm{H}^1_{\mathrm{dR}}(X_L,\mathscr{F}_L). \tag{1.25}$$

□

Lemma 1.4.8 ([?, Lemma 4.15]). *Assume that X has no proper connected component. Let W be an analytic domain of X such that the restriction map $\mathscr{O}(X) \to \mathscr{O}(W)$ has dense image. Assume that there exists a complete valued extension L of K with non-trivial valuation such that $\mathrm{H}^1_{\mathrm{dR}}(W_L, (\mathscr{F}_L)_{|W_L})$ is finite-dimensional. Then, the map*

$$\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(W, \mathscr{F}_{|W}) \tag{1.26}$$

is surjective. □

To finish this section, we recall that if X is a curve that can be conveniently "*approximated*" by a family of curves $\{X_n\}_n$, then the de Rham cohomology of the differential equation $\mathscr{F}$ over X can be recovered as the limit of the de Rham cohomologies of its restrictions to the X_n's.

This technique has been introduced by Christol and Mebkhout for open annuli (see [CM00, Proof of 8.3-1]) and it essentially follows from an original idea of Grothendieck (see [Gro61, Chap.0, 13.2.4] and even [Gro54]).

Theorem 1.4.9 ([?, Theorem 5.4]). *Assume that K is not trivially valued. Let X be a quasi-smooth K-analytic curve with finitely many connected components, none of them being proper. Assume moreover that there exists a non-decreasing sequence of analytic domains $(X_n)_{n\in\mathbb{N}}$ of X forming a covering of X for the G-topology and an integer n_0 such that, for every $n \geqslant n_0$,*

i) the natural map $\pi_0(X_n) \to \pi_0(X)$ is bijective;

ii) the restriction map $\mathscr{O}(X_{n+1}) \to \mathscr{O}(X_n)$ has dense image;

iii) the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n})$ is a finite-dimensional K-vector space.

Then,

(a) we have

$$\mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{F}) = \varprojlim_n \mathrm{H}^0_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}), \tag{1.27}$$

$\mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{F})$ is a finite-dimensional K-vector space and there exists an integer n_1 such that for every $n, m \in \mathbb{N}$ satisfying $n \geqslant m \geqslant n_1$, the natural map $\mathrm{H}^0_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}) \to \mathrm{H}^0_{\mathrm{dR}}(X_m, \mathscr{F}_{|X_m})$ is an isomorphism;

(b) for every $n, m \geqslant n_0$, the natural map $\mathrm{H}^1_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}) \to \mathrm{H}^1_{\mathrm{dR}}(X_m, \mathscr{F}_{|X_m})$ is surjective;

(c) one has

$$\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) = \varprojlim_n \mathrm{H}^1_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}) \tag{1.28}$$

and for every $n \geqslant n_0$, the natural map $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) \to \mathrm{H}^1_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n})$ is surjective.

In particular, $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F})$ is finite-dimensional if, and only if, the sequence of dimensions $(\mathrm{h}^1_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}))_{n\in\mathbb{N}}$ (or equivalently the sequence of indexes $(\chi_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}))_{n\in\mathbb{N}}$) is eventually constant. In this case, the natural map $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) \to \mathrm{H}^1_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n})$ is an isomorphism for all n large enough and we have

$$\mathrm{h}^1_{\mathrm{dR}}(X, \mathscr{F}) = \lim_{n\to+\infty} \mathrm{h}^1_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}) \quad \text{and} \quad \chi_{\mathrm{dR}}(X, \mathscr{F}) = \lim_{n\to+\infty} \chi_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}). \tag{1.29}$$

□

1.4.3. Sub-quotients In this section, we prove some results about sub-quotients and irreducible modules.

Lemma 1.4.10. *Assume that X is cohomologically Stein and has finitely many connected components.*

If $\mathscr{F}$ has finite-dimensional de Rham cohomology, then so has each sub-quotient of $\mathscr{F}$.

In particular, if $\mathrm{End}(\mathscr{F})$ *has finite-dimensional de Rham cohomology, then, for any two sub-quotients $\mathscr{F}_1$ and $\mathscr{F}_2$ of $\mathscr{F}$,* $\mathrm{Hom}(\mathscr{F}_1, \mathscr{F}_2)$ *has finite-dimensional de Rham cohomology.*

Proof. It is enough to show that, for each short exact sequence $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{H} \to 0$, if $\mathscr{F}$ has finite-dimensional de Rham cohomology, then so have $\mathscr{G}$ and $\mathscr{H}$. By Lemma 1.4.4, the groups $\mathrm{H}^0_{\mathrm{dR}}(X, -)$ and $\mathrm{H}^1_{\mathrm{dR}}(X, -)$ may be respectively computed as the kernel and the cokernel of the connection map. The snake lemma then provides the exact sequence

$$0 \to \mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{G}) \to \mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{F}) \to \mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{H}) \to \mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{G}) \to \mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) \to \mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{H}) \to 0. \tag{1.30}$$

All the $\mathrm{H}^0_{\mathrm{dR}}(X, -)$'s are finite-dimensional and so is $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F})$ by assumption. The result follows.

The second part of the statement follows from the first, using the fact that $\mathrm{Hom}(\mathscr{F}_1, \mathscr{F}_2) \cong \mathscr{F}_2^* \otimes \mathscr{F}_1$ is isomorphic to a sub-quotient of $\mathrm{End}(\mathscr{F}) \cong \mathscr{F}^* \otimes \mathscr{F}$. □

Lemma 1.4.11. *Assume that X is cohomologically Stein, that the trivial differential equation $\mathscr{O}$ has finite-dimensional de Rham cohomology and that $\chi_{\mathrm{dR}}(X, \mathscr{O}) \leqslant 0$. Assume that $\mathscr{F}$ has finite-dimensional de Rham cohomology. Then we have*

$$\chi_{\mathrm{dR}}(X, \mathscr{F}) \leqslant 0. \tag{1.31}$$

If, moreover, we have $\chi_{\mathrm{dR}}(X, \mathscr{F}) = 0$, then each sub-quotient $\mathscr{G}$ of $\mathscr{F}$ has finite-dimensional de Rham cohomology and satisfies $\chi_{\mathrm{dR}}(X, \mathscr{G}) = 0$.

Proof. First note that X has finitely many connected components since the dimension of $\mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{O})$ is equal to the cardinal of the set of connected components of X. By Lemma 1.4.4 and Lemma 1.4.5, it is enough to prove that the result holds for each irreducible sub-quotient $\mathscr{H}$ of $\mathscr{F}$.

Let us consider such a $\mathscr{H}$. By Lemma 1.4.10, it has finite-dimensional de Rham cohomology. If $\mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{H}) = 0$, then $\chi_{\mathrm{dR}}(X, \mathscr{H}) = -h^1(X, \mathscr{H}) \leqslant 0$. If $\mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{H}) \neq 0$, then $\mathscr{H}$ has a sub-module isomorphic to $\mathscr{O}$, hence $\mathscr{H}$ is isomorphic to $\mathscr{O}$ since it is irreducible by assumption. The result follows.

The second part of the result follows from the first using Lemma 1.4.5. □

Definition 1.4.12. *We say a module with connection is irreducible if it has no non-trivial sub-objects.*

We now prove a decomposition result for a certain class of modules which is similar to the structure of finite length modules (cf. [Bou12, p.28, Prop.3, §2]).

Lemma 1.4.13. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two differential modules over some quasi-smooth curve Y such that*

i) $\mathscr{G}_1$ and $\mathscr{G}_2$ have no common irreducible sub-quotient;

ii) $\chi_{\mathrm{dR}}(Y, \mathscr{G}_1^* \otimes \mathscr{G}_2) = 0$.

Then any exact sequence $0 \to \mathscr{G}_1 \to \mathscr{G} \to \mathscr{G}_2 \to 0$ splits.

Proof. By i) there are no non-zero homomorphisms $\mathscr{G}_1 \to \mathscr{G}_2$ commuting with the connections. Therefore $\mathrm{H}^0_{\mathrm{dR}}(Y, \mathrm{Hom}(\mathscr{G}_1, \mathscr{G}_2)) = 0$. Now, one has $\mathrm{Hom}(\mathscr{G}_1, \mathscr{G}_2) \cong \mathscr{G}_1^* \otimes \mathscr{G}_2$, therefore by ii) we also have $\mathrm{H}^1_{\mathrm{dR}}(Y, \mathscr{G}_1^* \otimes \mathscr{G}_2) = 0$. The claim then follows from the fact that $\mathrm{H}^1_{\mathrm{dR}}(Y, \mathscr{G}_1^* \otimes \mathscr{G}_2) \cong \mathrm{Ext}_{Diff.Eq.}(\mathscr{G}_2, \mathscr{G}_1)$ (cf. [Ked10, Lemma 5.3.3]). □

Proposition 1.4.14. *Let Y be a quasi-Stein quasi-smooth curve with a finite number of connected components. Let $\mathscr{F}$ be a differential equation over Y such that $\chi_{\mathrm{dR}}(Y, \mathrm{End}(\mathscr{F})) = 0$. Then*

$$\mathscr{F} \;=\; \bigoplus_i \mathscr{Q}_i \;, \tag{1.32}$$

where

i) for all i all irreducible sub-quotients of the differential equation $\mathscr{Q}_i$ are isomorphic each other;

ii) if $i \neq j$, then $\mathscr{Q}_i$ and $\mathscr{Q}_j$ have no common irreducible sub-quotients.

Proof. The category of differential equations over Y is Artinian and Jordan-Hölder theorem holds. Let $0 = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \ldots \subset \mathscr{M}_s = \mathscr{F}$ be a Jordan-Hölder sequence for $\mathscr{F}$, and let $\mathscr{N}_k = \mathscr{M}_k/\mathscr{M}_{k-1}$.

We call *block* a sub-quotient $\mathscr{Q}$ of $\mathscr{F}$ with the property that all the irreducible sub-quotients of $\mathscr{Q}$ are isomorphic each other. We say that two blocks are *similar* if their irreducible sub-quotients are isomorphic. Each Jordan-Hölder sequence of $\mathscr{F}$ furnishes a well-defined and ordered sequence of blocks $(\mathscr{Q}_1, \mathscr{Q}_2, \ldots, \mathscr{Q}_t)$ such that $\mathscr{Q}_j$ is not similar to $\mathscr{Q}_{j+1}$. More precisely, if $k_1 < \cdots < k_t \in \{1, \ldots, s\}$ is the set of indexes k such that $k = s$ or $\mathscr{N}_k \neq \mathscr{N}_{k+1}$, then $\mathscr{Q}_1, \ldots, \mathscr{Q}_t$ are the sub-quotients defined by $\mathscr{Q}_1 = \mathscr{M}_{k_1}$ and by $\mathscr{Q}_j = \mathscr{M}_{k_j}/\mathscr{M}_{k_{j-1}}$ for $j \in \{2, \ldots, t\}$.

It follows from the definition that two consecutive blocks are not similar, but this is not necessarily the case for arbitrary blocks, depending on the Jordan-Hölder sequence we started with. We want to prove that there exists a Jordan-Hölder sequence of $\mathscr{F}$ such that for any $j_1 \neq j_2$ the blocks $\mathscr{Q}_{j_1}$ and $\mathscr{Q}_{j_2}$ are not similar.

We claim that for any two sub-quotients $\mathscr{G}_1$ and $\mathscr{G}_2$ of $\mathscr{F}$ we have $\chi_{\mathrm{dR}}(Y, \mathscr{G}_1^* \otimes \mathscr{G}_2) = 0$. Indeed, notice that we can identify $\mathscr{G}_1^* \otimes \mathscr{G}_2$ with a sub-quotient of $\mathscr{F}^* \otimes \mathscr{F} \cong \mathrm{End}(\mathscr{F})$ and by Lemma 1.4.11 the vanishing of $\chi_{\mathrm{dR}}(Y, \mathrm{End}(\mathscr{F}))$ implies that of $\chi_{\mathrm{dR}}(Y, \mathscr{G}_1^* \otimes \mathscr{G}_2)$.

By Lemma 1.4.13, we see that for all $j = 1, \ldots, t-1$, the unique extensions between $\mathscr{Q}_j$ and $\mathscr{Q}_{j+1}$ is the direct sum. Therefore, $\mathscr{M}_{k_{j+1}}/\mathscr{M}_{k_{j-1}}$ admits a projection onto $\mathscr{Q}_j$ with kernel $\mathscr{Q}_{j+1}$. This shows that there exists a Jordan-Hölder sequence of $\mathscr{F}$ in which $\mathscr{Q}_j$ and $\mathscr{Q}_{j+1}$ are permuted (i.e. the block $\mathscr{Q}_j$ appears just after $\mathscr{Q}_{j+1}$).

In this way, we obtain a new Jordan-Hölder sequence for $\mathscr{F}$ and we can reinitialize the whole process and obtain new $\mathscr{Q}_j$'s. In general, two blocks $\mathscr{Q}_j$ and $\mathscr{Q}_{j'}$ can be permuted if the irreducible sub-quotients of $\mathscr{Q}_j$ are not isomorphic to those of $\mathscr{Q}_{j'}$, that is if they are not similar. Proceeding so, we can regroup the blocks that are similar each other in order to obtain a new Jordan-Hölder sequence for $\mathscr{F}$, and new $\mathscr{Q}_j$'s, with the desired property that for all $j \neq j'$, the block $\mathscr{Q}_j$ is not similar to $\mathscr{Q}_{j'}$. The claim then follows from Lemma 1.4.13. □

1.5. Meromorphic analytic and algebraic de Rham cohomologies.

In this section, we introduce definitions and basic results about algebraic and analytic meromorphic differential equations.

1.5.1. Meromorphic differential equations in the analytic setting.

Let P be a quasi-smooth K-analytic curve. Let Z be a locally finite subset of rigid points of P.

Set

$$Y \;:=\; P - Z \;. \tag{1.33}$$

We denote by $j\colon Y \hookrightarrow P$ the associated open immersion. We denote by $\mathscr{O}_P[*Z]$ the sheaf of meromorphic functions on P that are holomorphic on Y (hence have poles at worst on Z). Recall that it is the sheaf on P associated to the presheaf whose ring of sections on an analytic domain U of P is the localization of $\mathscr{O}_P(U)$ by the subset of its elements that do not vanish outside Z.

We now define meromorphic connections following [HTT08, Chapter 5] (which itself borrows from [Del70]).

Definition 1.5.1. *Let $\mathcal{F}$ be a locally free $\mathscr{O}_P[*Z]$-module of finite rank on P. A* meromorphic connection on $\mathcal{F}$ with poles on Z *is a K-linear map*

$$\nabla\colon \mathcal{F} \to \Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F} \tag{1.34}$$

that satisfies the Leibniz rule: for every open subset U of P and every $f \in \mathscr{O}_P[*Z](U)$ and $s \in \mathcal{F}(U)$, we have[2]

$$\nabla(fs) = df \otimes s + f\nabla s. \tag{1.35}$$

We also say that the pair $(\mathcal{F}, \nabla)$ is a differential equation on $P(*Z)$ *or a* (meromorphic) differential equation on P with poles on Z. *As usual, morphisms of differential equations $\varphi\colon (\mathcal{F}, \nabla) \to (\mathcal{F}', \nabla')$ are morphisms of $\mathscr{O}_P[*Z]$-modules that are compatible with the connections.*

Definition 1.5.2. *Let $(\mathcal{F}, \nabla)$ be a differential equation on $P(*Z)$. The de Rham cohomology groups*

$$\mathrm{H}^i_{\mathrm{dR}}(P(*Z), (\mathcal{F}, \nabla)) \tag{1.36}$$

of $(\mathcal{F}, \nabla)$ are the hypercohomology groups of the complex

$$\cdots \to 0 \to \mathcal{F} \xrightarrow{\nabla} \Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F} \to 0 \to \cdots, \tag{1.37}$$

where $\mathcal{F}$ is placed in degree 0 and $\Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F}$ in degree 1.

As usual, we will often suppress ∇ from the notation when it is clear from the context.

The notation $\mathscr{F}$ will be used to indicate the restriction of $\mathcal{F}$ to Y:

$$\mathscr{F} \;:=\; \mathcal{F}_{|Y}\,. \tag{1.38}$$

This operation gives rise to a functor

$$\begin{array}{c}\{\text{Differential equations on } P(*Z)\}\\ \Big\downarrow{\scriptstyle \mathcal{F} \mapsto \mathcal{F}_{|Y} = \mathscr{F}}\\ \{\text{Analytic differential equations on } Y\}\end{array} \tag{1.39}$$

and a canonical morphism between the cohomology groups

$$\mathrm{H}^i_{\mathrm{dR}}(P(*Z), \mathcal{F}) \;\to\; \mathrm{H}^i_{\mathrm{dR}}(Y, \mathscr{F})\,. \tag{1.40}$$

When we do not mention poles, we understand that the connection is holomorphic: $Z = \emptyset$ and $\mathcal{F} = \mathscr{F}$ is a genuine analytic differential equation over $Y = P$.

Remark 1.5.3. *If U is an open subset of P such that $U \cap Z = \emptyset$, the restriction of the sheaf $\mathscr{O}_P[*Z]$ to U coincides by definition with $\mathscr{O}_U$. Hence, over U, we have the usual* analytic *cohomology:*

$$\mathrm{H}^i_{\mathrm{dR}}(U(*Z), \mathcal{F}_{|U}) \;=\; \mathrm{H}^i_{\mathrm{dR}}(U, \mathscr{F}_{|U})\,. \tag{1.41}$$

Lemma 1.5.4. *For each locally free $\mathscr{O}_P[*Z]$-module of finite rank $\mathcal{F}$ on P, there exists a locally free $\mathscr{O}_P$-module of finite rank G such that we have an isomorphism of $\mathscr{O}_P[*Z]$-modules (without connections) $G \otimes_{\mathscr{O}_P} \mathscr{O}_P[*Z] \simeq \mathcal{F}$.*

Proof. Every point z of Z admits a neighborhood U_z of z on which the restriction of $\mathcal{F}$ is isomorphic to $\mathscr{O}_P[*\{z\}]^d$ for some d. In particular, it is isomorphic to $\mathscr{O}_P^d$ over $U_z - \{z\}$, hence extends to $\mathscr{O}_P^d$ over U_z. □

[2] Remark that it is enough to require that (1.35) holds for $f \in \mathscr{O}_P(U)$.

Lemma 1.5.5 ([?, Lemma 4.17])**.** *Let W be an analytic domain of Y such that the restriction map $(\mathscr{O}_P[*Z])(P) \to \mathscr{O}(W)$ has dense image. Assume that there exists a complete valued extension L of K with non-trivial valuation such that $\mathrm{H}^1_{\mathrm{dR}}(W_L, \mathscr{F}_L)$ is finite-dimensional. Then the map*

$$\mathrm{H}^1_{\mathrm{dR}}(P(*Z), \mathcal{F}) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(W, \mathscr{F}) \tag{1.42}$$

is surjective. □

1.5.2. Meromorphic differential equations in the algebraic setting. Here, we turn to the algebraic setting and consider schemes endowed with the Zariski topology. We similarly define the notion of meromorphic differential equation $(\mathfrak{F}, \nabla)$ (and meromorphic connection) over a smooth algebraic curve $\mathfrak{P}$ with poles on a locally finite subset of closed points $\mathfrak{Z}$ and the associated de Rham cohomology groups.

If $P = \mathfrak{P}^{\mathrm{an}}$ and $Z = \mathfrak{Z}^{\mathrm{an}}$, we have functors

$$\begin{array}{c} \{\text{Algebraic diff. eq. on } \mathfrak{P} \text{ with poles on } \mathfrak{Z}\} \\ \Big\downarrow {\scriptstyle \mathfrak{F} \mapsto \mathfrak{F}^{\mathrm{an}}} \\ \{\text{Differential equations on } P(*Z)\} \\ \Big\downarrow {\scriptstyle \mathcal{F} \mapsto \mathcal{F}_{|Y} = \mathscr{F}} \\ \{\text{Analytic differential equations on } Y\}\ . \end{array} \tag{1.43}$$

and canonical morphisms between the cohomology groups

$$\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}(*\mathfrak{Z}), \mathfrak{F}) \to \mathrm{H}^i_{\mathrm{dR}}(P(*Z), \mathfrak{F}^{\mathrm{an}})\ , \tag{1.44}$$

$$\mathrm{H}^i_{\mathrm{dR}}(P(*Z), \mathcal{F}) \to \mathrm{H}^i_{\mathrm{dR}}(Y, \mathcal{F}_{|Y})\ . \tag{1.45}$$

1.5.3. Differential operators of order one. Meromorphic connections may also be defined using differential operators of order 1, following [Gro67, §16.7 and §16.8]. Since this is useful for our purposes, we quickly recall it now.

Denote by $\delta \colon P \mapsto P \times P$ the diagonal embedding. The map $\delta^*(\mathscr{O}_{P\times P}) \to \mathscr{O}_P$ is surjective and we denote its kernel by $\mathscr{I}$. Define the first infinitesimal neighborhood $P^{(1)}$ of P to be $(P, \delta^*(\mathscr{O}_{P\times P})/\mathscr{I}^2)$. We have two morphisms $p_1^{(1)}, p_2^{(1)} \colon P^{(1)} \to P$ defined using the two projections p_1 and p_2 from $P \times P$ to P.

Let $\mathcal{F}$ be a locally free $\mathscr{O}_P[*Z]$-module of finite rank on P. Set

$$\mathscr{P}^{(1)}(\mathcal{F}) := (p_1^{(1)})_*(p_2^{(1)})^*\mathcal{F} \tag{1.46}$$

and endow it with the structure of $\mathscr{O}_P$-module induced by $p_1^{(1)}$. The other map $p_2^{(1)}$ gives rise to a morphism of sheaves of groups

$$d^1_{\mathcal{F}} \colon \mathcal{F} \to \mathscr{P}^{(1)}(\mathcal{F}). \tag{1.47}$$

Note that, locally on P, we can write $\mathscr{O}_{P^{(1)}} = (\mathscr{O}_P \hat{\otimes}_K \mathscr{O}_P)/\mathscr{I}^2$ and

$$\mathscr{P}^{(1)}(\mathscr{F}) = \mathscr{O}_{P^{(1)}} \otimes_{\mathscr{O}_P} \mathcal{F} \tag{1.48}$$

where the tensor product is formed considering the $\mathscr{O}_P$-structure on $\mathscr{O}_{P^{(1)}}$ obtained via $p_2^{(1)}$. Again, the structure of $\mathscr{O}_P$-module is induced by $p_1^{(1)}$. The map $d^1_{\mathcal{F}}$ is then locally given by $d^1_{\mathcal{F}}(x) = 1 \otimes x$.

Note that it is also possible to define $\mathscr{P}^{(1)}(\mathscr{O}_P)$ first and then set

$$\mathscr{P}^{(1)}(\mathcal{F}) := \mathscr{P}^{(1)}(\mathscr{O}_P) \otimes_{\mathscr{O}_P} \mathcal{F}, \tag{1.49}$$

where the map $\mathscr{O}_P \to \mathscr{P}^{(1)}(\mathscr{O}_P)$ used in the tensor product is $d^1_{\mathscr{O}_P}$.

Definition 1.5.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathscr{O}_P[*Z]$-modules. A morphism of sheaves of groups $D\colon \mathcal{F} \to \mathcal{G}$ is said to be a differential operator of order 1 if there exists a morphism of $\mathscr{O}_P$-modules $u\colon \mathscr{P}^{(1)}(\mathcal{F}) \to \mathcal{G}$ such that $D = u \circ d^1_{\mathcal{F}}$.*

The relation between differential operators of order 1 and meromorphic connections is made precise by the following result. We leave the proof to the reader (who can find it, in the complex analytic setting, at the end of the proof of [HTT08, Theorem 5.3.8]).

Lemma 1.5.7. *Meromorphic connection structures on the $\mathscr{O}_P[*Z]$-module $\mathcal{F}$ in the sense of Definition 1.5.1 correspond bijectively (via composition by $d^1_{\mathcal{F}}$) to global sections on P of the sheaf $\mathscr{H}om_{\mathscr{O}_P}(\mathscr{P}^{(1)}(\mathcal{F}), \Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F})$ that induce the identity on*

$$\Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F} = (\mathscr{I}/\mathscr{I}^2) \otimes_{\mathscr{O}_P} \mathcal{F} \subset \mathscr{P}^{(1)}(\mathscr{O}_P) \otimes_{\mathscr{O}_P} \mathcal{F} = \mathscr{P}^{(1)}(\mathcal{F}) \tag{1.50}$$

(see (1.49) for the last equality). □

Of course, the previous construction also works in the algebraic setting.

1.5.4. Comparison results: the projective case. Let us now compare the algebraic and analytic settings. Here, we follow [Del70, II, §6] and also borrow from the proof of [HTT08, Theorem 5.3.8].

Let $\mathfrak{P}$ be a smooth algebraic curve over K and let $\mathfrak{Z}$ be a locally finite subset of closed points of $\mathfrak{P}$. Let $\mathfrak{F}$ be a locally free $\mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}]$-module of finite rank and let ∇ be a meromorphic connection structure on $\mathfrak{F}$. We may write ∇ as a composition

$$\nabla\colon \mathfrak{F} \xrightarrow{d^1_{\mathfrak{F}}} \mathscr{P}^{(1)}(\mathfrak{F}) \xrightarrow{u} \Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathfrak{F}, \tag{1.51}$$

where u is $\mathscr{O}_{\mathfrak{P}}$-linear.

Denote by P (resp. Z) the analytification of $\mathfrak{P}$ (resp. $\mathfrak{Z}$). Remark that the analytification functor for $\mathscr{O}_{\mathfrak{P}}$-modules behaves well with respect to $\mathscr{P}^{(1)}$ in the sense that $(\mathscr{P}^{(1)}(\mathfrak{F}))^{\mathrm{an}} = \mathscr{P}^{(1)}(\mathfrak{F}^{\mathrm{an}})$. Note also that $\mathfrak{F}^{\mathrm{an}}$ is naturally an $\mathscr{O}_P[*Z]$-module. We may now define the analytification of the meromorphic connection ∇ by

$$\nabla^{\mathrm{an}}\colon \mathfrak{F}^{\mathrm{an}} \xrightarrow{d^1_{\mathfrak{F}^{\mathrm{an}}}} \mathscr{P}^{(1)}(\mathfrak{F})^{\mathrm{an}} \xrightarrow{u^{\mathrm{an}}} \Omega^1_P \otimes_{\mathscr{O}_P} \mathfrak{F}^{\mathrm{an}}. \tag{1.52}$$

Recall that the GAGA theorems hold in the rigid analytic setting, starting from proper schemes over a complete valued field or even over an affinoid space (see [Köp74] or [Poi10, Annexe A] for a proof in the setting of Berkovich spaces). The usual version deals with coherent sheaves but the following one is an easy consequence of it (see [Del70, II, Lemme 6.5] for details in the complex analytic case).

Theorem 1.5.8. *Let $\mathfrak{P}$ be a projective algebraic variety over K. Let $\mathfrak{F}$ be a sheaf of $\mathscr{O}_{\mathfrak{P}}$-modules that is a filtered direct limit of coherent sheaves. Then, for every $i \geqslant 0$, the natural map*

$$H^i(\mathfrak{P}, \mathfrak{F}) \to H^i(\mathfrak{P}^{\mathrm{an}}, \mathfrak{F}^{\mathrm{an}}) \tag{1.53}$$

is an isomorphism. □

Corollary 1.5.9. *Let $\mathfrak{P}$ be a smooth projective algebraic curve over K and $\mathfrak{Z}$ be a locally finite subset of closed points of $\mathfrak{P}$. Let $\mathfrak{F}$ be a locally free $\mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}]$-module endowed with a meromorphic connection ∇ and let $\mathcal{F} := \mathfrak{F}^{\mathrm{an}}$. Then, for every $i \geqslant 0$, we have a natural isomorphism*

$$\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}(*\mathfrak{Z}), \mathfrak{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(P(*Z), \mathcal{F})\ . \tag{1.54}$$

Proof. Let $\mathfrak{D}$ denote the de Rham complex

$$\cdots \to 0 \to \mathfrak{F} \xrightarrow{\nabla} \Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathfrak{F} \to 0 \to \cdots \tag{1.55}$$

The de Rham cohomology of $(\mathfrak{F}, \nabla)$ is the hypercohomology of $\mathfrak{D}$ and may be computed by the spectral sequence

$$E_1^{pq} = H^p(\mathfrak{P}, \mathfrak{D}^q) \implies \mathrm{H}^{p+q}_{\mathrm{dR}}(\mathfrak{P}(*\mathfrak{Z}), \mathfrak{F}). \tag{1.56}$$

Similarly, on the analytic side, we have

$$E_1^{pq} = H^p(P, \mathfrak{D}^{q,\mathrm{an}}) \implies \mathrm{H}^{p+q}_{\mathrm{dR}}(P(*Z), \mathfrak{F}^{\mathrm{an}}). \tag{1.57}$$

Since the sheaves $\mathfrak{D}^q$ (*i.e.* $\mathfrak{F}$ for $q = 0$, $\Omega^1_{\mathfrak{P}} \otimes \mathfrak{F}$ for $q = 1$ and 0 otherwise) are filtered direct limits of coherent sheaves, we may apply Theorem 1.5.8 to the canonical morphism between the first pages of the two spectral sequences and conclude that it is an isomorphism. The result follows.

□

Let us now compare the algebraic and analytic categories of modules with meromorphic connections.

Proposition 1.5.10. *Let $\mathfrak{P}$ be a smooth projective algebraic curve over K and $\mathfrak{Z}$ be a locally finite subset of closed points of $\mathfrak{P}$. The functor $(\mathfrak{F}, \nabla) \mapsto (\mathfrak{F}^{\mathrm{an}}, \nabla^{\mathrm{an}})$ sets up an equivalence between the category of locally free $\mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}]$-modules of finite rank endowed with a meromorphic connection and the category of locally free $\mathscr{O}_{\mathfrak{P}^{\mathrm{an}}}[\mathfrak{Z}^{\mathrm{an}}]$-modules of finite rank endowed with a meromorphic connection.*

Proof. Let us prove essential surjectivity first. Let $P := \mathfrak{P}^{\mathrm{an}}$, and $Z = \mathfrak{Z}^{\mathrm{an}}$, and let $\mathcal{F}$ be a locally free $\mathscr{O}_P[*Z]$-module of finite rank endowed with a meromorphic connection ∇'. By Lemma 1.5.4, there exists a locally free $\mathscr{O}_P$-module of finite rank G such that $G \otimes_{\mathscr{O}_P} \mathscr{O}_P[*Z] \simeq \mathcal{F}$. By GAGA, there exists a locally free $\mathscr{O}_{\mathfrak{P}}$-module of finite rank $\mathfrak{G}$ such that $\mathfrak{G}^{\mathrm{an}} \simeq G$. If we set $\mathfrak{F} := \mathfrak{G} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathscr{O}_{\mathfrak{P}}[*Z]$, we have an isomorphism of $\mathscr{O}_P[*Z]$-modules $\mathfrak{F}^{\mathrm{an}} \simeq \mathcal{F}$. We will now identify $\mathfrak{F}^{\mathrm{an}}$ and $\mathcal{F}$.

It remains to prove that there exists a meromorphic connection ∇ on $\mathfrak{F}$ such that $\nabla^{\mathrm{an}} = \nabla'$. By Lemma 1.5.7, ∇' is a global section on $P = \mathfrak{P}^{\mathrm{an}}$ of

$$\mathscr{H}om_{\mathscr{O}_P}(\mathscr{P}^{(1)}(\mathcal{F}), \Omega^1_P \otimes_{\mathscr{O}_P} \mathcal{F}) = (\mathscr{H}om_{\mathscr{O}_{\mathfrak{P}}}(\mathscr{P}^{(1)}(\mathfrak{F}), \Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathfrak{F}))^{\mathrm{an}}. \tag{1.58}$$

Since the sheaf $\mathscr{H}om_{\mathscr{O}_{\mathfrak{P}}}(\mathscr{P}^{(1)}(\mathfrak{F}), \Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathfrak{F})$ is a filtered direct limit of coherent sheaves on $\mathfrak{P}$, we may use Theorem 1.5.8 for H^0. This shows that ∇' corresponds to a global section of the sheaf $\mathscr{H}om_{\mathscr{O}_{\mathfrak{P}}}(\mathscr{P}^{(1)}(\mathfrak{F}), \Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathfrak{F})$ on $\mathfrak{P}$ (whose restriction to $\Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathfrak{F} \subset \mathscr{P}^{(1)}(\mathfrak{F})$ is equal to the identity), *i.e.* a meromorphic connection ∇ on $\mathfrak{F}$.

Let us now prove that the functor is fully faithful. Let $(\mathfrak{F}_1, \nabla_1)$ and $(\mathfrak{F}_2, \nabla_2)$ be locally free $\mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}]$-modules endowed with meromorphic connections. It is enough to prove that the canonical map

$$\mathrm{Hom}_{\mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}]}(\mathfrak{F}_1, \mathfrak{F}_2) \to \mathrm{Hom}_{\mathscr{O}_P[*Z]}(\mathfrak{F}_1^{\mathrm{an}}, \mathfrak{F}_2^{\mathrm{an}}) \tag{1.59}$$

is a bijection. By arguing as in the proof of Lemma 1.5.4 in the algebraic setting, we prove that there exists a locally free $\mathscr{O}_{\mathfrak{P}}$-submodule F_1 of $\mathfrak{F}_1$ such that $F_1 \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}] \simeq \mathfrak{F}_1$. We have bijective morphisms

$$\mathrm{Hom}_{\mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}]}(\mathfrak{F}_1, \mathfrak{F}_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_{\mathfrak{P}}}(F_1, \mathfrak{F}_2)\ , \tag{1.60}$$

$$\mathrm{Hom}_{\mathscr{O}_P[*Z]}(\mathfrak{F}_1^{\mathrm{an}}, \mathfrak{F}_2^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_P}(F_1^{\mathrm{an}}, \mathfrak{F}_2^{\mathrm{an}})\ . \tag{1.61}$$

Moreover, $\mathfrak{F}_2$ may be written as a direct limit of a family $(\mathfrak{F}_{2,\lambda})_{\lambda \in \Lambda}$ of coherent $\mathscr{O}_{\mathfrak{P}}$-submodules, hence

$$\begin{aligned}\mathrm{Hom}_{\mathscr{O}_{\mathfrak{P}}}(F_1, \mathfrak{F}_2) &\simeq \varinjlim_{\lambda\in\Lambda} \mathrm{Hom}_{\mathscr{O}_{\mathfrak{P}}}(F_1, \mathfrak{F}_{2,\lambda}) && (1.62)\\ &\simeq \varinjlim_{\lambda\in\Lambda} \mathrm{Hom}_{\mathscr{O}_P}(F_1^{\mathrm{an}}, \mathfrak{F}_{2,\lambda}^{\mathrm{an}}) && (1.63)\\ &\simeq \mathrm{Hom}_{\mathscr{O}_P}(F_1^{\mathrm{an}}, \mathfrak{F}_2^{\mathrm{an}}), && (1.64)\end{aligned}$$

where the first and third bijections comes from the fact that F_1 and F_1^{an} are modules of finite type and the second bijection from GAGA. The result follows. □

Remark 1.5.11. *Assume that K is trivially valued. In this case, the GAGA theorems hold for arbitrary algebraic varieties over K, with no projectivity assumption (see [Ber90, Theorem 3.5.1]). We deduce that Corollary 1.5.9 and Proposition 1.5.10 hold for an arbitrary smooth algebraic curve $\mathfrak{P}$.*

1.5.5. Comparison results: the Stein case. Over a Stein space, it is natural to expect that one can read the properties of a sheaf on its global sections. We prove comparison results in this direction. The strategy is very close to that of Section 1.5.4.

Using the fact that cohomology and tensor products commute with filtered direct limits, we can generalize Theorem 1.2.2 in the following way.

Theorem 1.5.12. *Let P be a quasi-smooth K-analytic curve with no proper connected component. Let $\mathcal{F}$ be a sheaf of $\mathscr{O}_P$-modules that is a filtered direct limit of coherent sheaves. Then, for every $g \geqslant 1$, we have $H^q(P, \mathcal{F}) = 0$ and $\mathcal{F}$ is generated by its global sections.* □

Corollary 1.5.13. *In the setting of Theorem 1.5.12, the global section functor $\mathcal{F} \mapsto \mathcal{F}(P)$ sets up an equivalence between the category of $\mathscr{O}_P$-modules that are filtered direct limits of coherent sheaves and the category of $\mathscr{O}(P)$-modules.* □

We now pass to connections.

Corollary 1.5.14. *Let P be a quasi-smooth K-analytic curve with no proper connected component. Let Z be a locally finite subset of rigid points of P.*

The global section functor $\mathcal{F} \mapsto \mathcal{F}(P)$ sets up an equivalence between the category of locally free $\mathscr{O}_P[*Z]$-modules of bounded rank endowed with a meromorphic connection and the category of projective $(\mathscr{O}_P[*Z])(P)$-modules of finite type endowed with a connection.

Proof. Let $\mathcal{F}$ be a locally free $\mathscr{O}_P[*Z]$-module of finite rank. It is a filtered direct limit of coherent sheaves. Then, thanks to Theorem 1.5.12, we can adapt the arguments of the proof of Corollary 1.2.3 in order to show that $\mathcal{F}(P)$ is an $(\mathscr{O}_P[*Z])(P)$-module of finite type. Using the same arguments as in the proof of Corollary 1.2.4, we prove that it is projective.

Let us now prove essential surjectivity. Let M be a projective $(\mathscr{O}_P[*Z])(P)$-module of finite type endowed with a connection ∇_M. Denote by $\mathcal{F}$ the sheaf of $\mathscr{O}_P$-modules generated by M. It is a sheaf of $\mathscr{O}_P[*Z]$-modules of bounded rank and, by Corollary 1.5.13, we have $\mathcal{F}(P) = M$. Using the same arguments as in the proof of Corollary 1.2.4, we prove that $\mathcal{F}$ is locally free. Remark also that ∇_M induces a meromorphic connection ∇ on $\mathcal{F}$ (which obviously satisfies $\nabla(P) = \nabla_M$).

Finally, since locally free $\mathscr{O}_P[*Z]$-modules of bounded rank are generated by their global sections, the global section functor is fully faithful. □

1.5.6. Algebraic vs. meromorphic index formulas. Following [Del70], we now recall how to compute the Euler characteristic in the sense of de Rham cohomology of a locally free sheaf endowed with a meromorphic connection.

Let $\mathfrak{P}$ be a smooth geometrically connected projective algebraic curve over K and $\mathfrak{Z}$ be a locally finite subset of closed points of $\mathfrak{P}$. Set

$$\mathfrak{Y} \;:=\; \mathfrak{P} - \mathfrak{Z}\,. \tag{1.65}$$

Let $\mathfrak{F}$ be a locally free $\mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}]$-module endowed with a meromorphic connection ∇.

As in section 1.5.4, we set

$$P \;:=\; \mathfrak{P}^{\mathrm{an}}\,, \qquad Z \;:=\; \mathfrak{Z}^{\mathrm{an}}\,, \qquad Y \;:=\; \mathfrak{Y}^{\mathrm{an}}\,, \qquad \mathcal{F} \;:=\; \mathfrak{F}^{\mathrm{an}}\,. \tag{1.66}$$

Proposition 1.5.15. *Assume that $\mathfrak{P}$ is connected. For every $i \geqslant 0$, we have a natural isomorphism*

$$\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}(*\mathfrak{Z}),\mathfrak{F}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{Y},\mathfrak{F}_{|\mathfrak{Y}}). \tag{1.67}$$

Proof. We follow [Del70, II, §6.4]. Denote by j the inclusion $j\colon \mathfrak{Y}\to\mathfrak{P}$. Since the morphism j is affine, for every $i \geqslant 1$ and every quasi-coherent sheaf $\mathfrak{G}$ on $\mathfrak{Y}$ we have $R^i j_*\mathfrak{G} = 0$. We deduce that, for every $i \geqslant 0$, we have a natural isomorphism between the i^{th} cohomology group of the complex

$$\cdots \to 0 \to j_*(\mathfrak{F}_{|Y}) \xrightarrow{j_*\nabla_{|\mathfrak{Y}}} j_*(\Omega^1_{\mathfrak{Y}} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathfrak{F}_{|Y}) \to 0 \to \cdots \tag{1.68}$$

and that of the complex

$$\cdots \to 0 \to \mathfrak{F}_{|\mathfrak{Y}} \xrightarrow{\nabla_{|\mathfrak{Y}}} \Omega^1_{\mathfrak{Y}} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathfrak{F}_{|\mathfrak{Y}} \to 0 \to \cdots\,. \tag{1.69}$$

It is now enough to identify the complex (1.68) with

$$\cdots \to 0 \to \mathfrak{F} \xrightarrow{\nabla} \Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathfrak{F} \to 0 \to \cdots\,. \tag{1.70}$$

Since $\mathfrak{F}$ is an $\mathscr{O}_{\mathfrak{P}}[\mathfrak{Z}]$-module, we have $\mathfrak{F} \simeq j_*(\mathfrak{F}_{|\mathfrak{Y}})$. Moreover, by the projection formula, we have

$$j_*(\Omega^1_{\mathfrak{Y}} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathfrak{F}_{|Y}) \;\simeq\; j_*(j^*\Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathfrak{F}_{|Y}) \;\simeq\; \Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} j_*\mathfrak{F}_{|Y} \;\simeq\; \Omega^1_{\mathfrak{P}} \otimes_{\mathscr{O}_{\mathfrak{P}}} \mathfrak{F}\,. \tag{1.71}$$

The result follows. □

Remark 1.5.16. *The analytic analogue of Proposition 1.5.15 requires certain Liouville conditions at the germs of segments out of the points of Z to hold (see Corollary ??).*

Let M be a $K((T))$-differential module. Following [Del70, p.110] and [Mal74], from the T-adic valuation of the coefficients of an operator associated to M in a cyclic basis, one can construct the so-called formal Newton polygon of M. The total height $i_0(\mathrm{M})$ of that polygon is called the formal irregularity of M (see [PP13, Section 5.7] for more details).

More generally, let $z \in \mathfrak{Z}$. Consider the scalar extension $\mathfrak{P}' := \mathfrak{P} \otimes_K K(z)$ and denote by $\mathfrak{F}'$ the pull-back of $\mathfrak{F}$ to $\mathfrak{P}'$. Choose a $K(z)$-rational point z' of $\mathfrak{P}'$ over z. In this case, $\mathscr{O}_{\mathfrak{P}',z'}$ is a discrete valuation ring with maximal ideal generated by a local parameter $T_{z'}$. Its completion $\widehat{\mathscr{O}_{\mathfrak{P}',z'}}$ is isomorphic to the field of power series $K(z)((T_{z'}))$ with coefficients in $K(z)$. We define

$$i_z(\mathfrak{F}) \;:=\; i_0(\mathrm{M}_{z'})\cdot[K(z):K], \tag{1.72}$$

where $i_0(\mathrm{M}_{z'})$ denotes the formal irregularity of the differential module $\mathrm{M}_{z'} := \mathfrak{F}' \otimes_{\mathscr{O}_{\mathfrak{P}',z'}} K(z)((T_{z'}))$.

The definition is compatible with scalar extension in the following sense. Let L be a finite extension of K and consider the projection map $\pi_L\colon \mathfrak{P}\otimes_K L \to \mathfrak{P}$. Then, for every $z \in \mathfrak{Z}$, we have

$$i_z(\mathfrak{F}) = \sum_{\zeta\in\pi_L^{-1}(z)} i_\zeta(\pi_L^*(\mathfrak{F}))\,. \tag{1.73}$$

Proposition 1.5.17. *For every $i \geqslant 0$, the K-vector space $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}(*\mathfrak{Z}),\mathfrak{F}) \simeq \mathrm{H}^i_{\mathrm{dR}}(P(*Z),\mathcal{F})$ is finite-dimensional. Moreover, we have*

$$\chi_{\mathrm{dR}}(P(*Z),\mathcal{F}) \;=\; \chi_{\mathrm{dR}}(\mathfrak{P}(*\mathfrak{Z}),\mathfrak{F}) \;=\; \chi_c(\mathfrak{Y})\,\mathrm{rank}(\mathfrak{F}) - \sum_{z\in\mathfrak{Z}} i_z(\mathfrak{F})\;, \tag{1.74}$$

where $\chi_c(\mathfrak{Y}) = 2-2g(\mathfrak{P}) - \sum_{z\in\mathfrak{Z}}[K(z):K]$ (see [PP13, Definition 1.1.46]).

Proof. Let K_0 be a subfield of K finitely generated over $\mathbb{Q}$ such that there exists a smooth connected projective algebraic curve $\mathfrak{P}_0$ over K_0, a locally finite subset of closed points $\mathfrak{Z}_0$ of $\mathfrak{P}_0$ and a locally free $\mathscr{O}_{\mathfrak{P}_0}[\mathfrak{Z}_0]$-module endowed with a meromorphic connection ∇_0 such that $\mathfrak{P}_0\otimes_{K_0}K\simeq\mathfrak{P}$, $\mathfrak{Z}_0\otimes_{K_0}K\simeq\mathfrak{Z}$, $\pi^*_{K/K_0}(\mathfrak{F}_0)\simeq\mathfrak{F}$ and $\pi^*_{K/K_0}(\nabla_0)\simeq\nabla$, where $\pi_{K/K_0}\colon\mathfrak{P}_0\otimes_{K_0}K\to\mathfrak{P}_0$ is the base-change morphism. Since the tensor product by a field is faithfully flat, for every $i\geqslant 0$, we have a natural isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}_0(*\mathfrak{Z}_0),\mathfrak{F}_0)\otimes_{K_0}K \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}(*\mathfrak{Z}),\mathfrak{F}). \tag{1.75}$$

Remark that K_0 may be embedded into $\mathbb{C}$. We fix such an embedding and denote by $\pi_{\mathbb{C}/K_0}\colon\mathfrak{P}_0\otimes_{K_0}\mathbb{C}\to\mathfrak{P}_0$ the base-change morphism. Set $\mathfrak{P}_{\mathbb{C}} := \mathfrak{P}_0\otimes_{K_0}\mathbb{C}$, $\mathfrak{Z}_{\mathbb{C}} := \mathfrak{Z}_0\otimes_{K_0}\mathbb{C}$, $\mathfrak{F}_{\mathbb{C}} := \pi^*_{\mathbb{C}/K_0}(\mathfrak{F}_0)$ and $\nabla_{\mathbb{C}} := \pi^*_{\mathbb{C}/K_0}(\nabla_0)$. As before, for every $i\geqslant 0$, we have a natural isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}_0(*\mathfrak{Z}),\mathfrak{F}_0)\otimes_{K_0}\mathbb{C} \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}_{\mathbb{C}}(*Z),\mathfrak{F}_{\mathbb{C}}). \tag{1.76}$$

By Proposition 1.5.15, we have an isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{P}_{\mathbb{C}}(*Z),\mathfrak{F}_{\mathbb{C}}) \;\simeq\; \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{Y}_{\mathbb{C}},(\mathfrak{F}_{\mathbb{C}})_{|\mathfrak{Y}_{\mathbb{C}}}) \tag{1.77}$$

where $\mathfrak{Y}_{\mathbb{C}} := \mathfrak{P}_{\mathbb{C}} - \mathfrak{Z}_{\mathbb{C}}$. The finiteness of the dimension of the cohomology spaces now follows from [Del70, II, Proposition 6.20 (i)] and Corollary 1.5.9.

To conclude, it is enough to compute the Euler characteristic of $(\mathfrak{F}_{\mathbb{C}},\nabla_{\mathbb{C}})_{|\mathfrak{Y}_{\mathbb{C}}}$. By [Del70, II, (6.21.1)], we have

$$\chi_{\mathrm{dR}}(\mathfrak{Y}_{\mathbb{C}},(\mathfrak{F}_{\mathbb{C}})_{|\mathfrak{Y}_{\mathbb{C}}}) \;=\; \chi_{\mathrm{dR}}(\mathfrak{Y}^{\mathrm{an}}_{\mathbb{C}},(\mathfrak{F}^{\mathrm{an}}_{\mathbb{C}})_{|\mathfrak{Y}^{\mathrm{an}}_{\mathbb{C}}}) - \sum_{z\in\mathfrak{Z}_{\mathbb{C}}} i_z(\mathfrak{F}_{\mathbb{C}})\;, \tag{1.78}$$

where $(\cdot)^{\mathrm{an}}$ here denotes complex analytification. Since the irregularities are compatible with scalar extension in the sense of formula (1.73), we have

$$\sum_{z\in\mathfrak{Z}_{\mathbb{C}}} i_z(\mathfrak{F}_{\mathbb{C}}) = \sum_{z\in\mathfrak{Z}_0} i_z(\mathfrak{F}_0) = \sum_{z\in\mathfrak{Z}} i_z(\mathfrak{F})\;. \tag{1.79}$$

Finally, we have

$$\chi_{\mathrm{dR}}(\mathfrak{Y}^{\mathrm{an}}_{\mathbb{C}},(\mathfrak{F}^{\mathrm{an}}_{\mathbb{C}})_{|\mathfrak{Y}^{\mathrm{an}}_{\mathbb{C}}}) \;=\; \chi(\mathfrak{Y}^{\mathrm{an}}_{\mathbb{C}})\,\mathrm{rank}(\mathfrak{F}), \tag{1.80}$$

where $\chi(\mathfrak{Y}^{\mathrm{an}}_{\mathbb{C}})$ denotes the Euler characteristic of the complex analytic space $\mathfrak{Y}^{\mathrm{an}}_{\mathbb{C}}$. The latter space obtained from the Riemann surface $\mathfrak{P}^{\mathrm{an}}_{\mathbb{C}}$by removing the points of $\mathfrak{Z}^{\mathrm{an}}_{\mathbb{C}}$, so we have

$$\chi(\mathfrak{Y}^{\mathrm{an}}_{\mathbb{C}}) = 2-2g(\mathfrak{P}^{\mathrm{an}}_{\mathbb{C}}) - \mathrm{Card}(\mathfrak{Z}^{\mathrm{an}}_{\mathbb{C}}(\mathbb{C})) = 2-2g(\mathfrak{P}) - \sum_{z\in\mathfrak{Z}}[K(z):K]. \tag{1.81}$$

This concludes the proof. □

Remark 1.5.18. *Let $\mathscr{F} := \mathcal{F}_{|Y}$. In Section* **??**, *we will prove that the above index formula (1.74) can be rewritten using the analytic irregularities $\mathrm{Irr}_z(\mathscr{F})$ of $\mathscr{F}$ at the points of Z (see Definition* **??** *and Corollary* **??***):*

$$\chi_{\mathrm{dR}}(P(*Z),\mathcal{F}) \;=\; \chi_c(Y)\cdot\mathrm{rank}(\mathcal{F}) + \sum_{z\in Z}\mathrm{Irr}_z(\mathscr{F})\;. \tag{1.82}$$

Notice that $g(Y) = g(P)$ *and* $\chi_c(Y) = \chi_c(P) - \sum_{z\in Z}[\mathscr{H}(z) : K]$.

Remark 1.5.19. *An interesting feature of the previous result is that, in this setting, the finite-dimensionality of the meromorphic de Rham cohomology spaces* $\mathrm{H}^i_{\mathrm{dR}}(P(*Z),\mathcal{F})$ *holds* without any Liouville assumptions. *This fact is used in the proof of the index result of [CM01] and has also been recently put in light by Kedlaya (see [Ked15a, Definition 7.5 and Lemma 7.6], which drew our attention to it).*

The index formula (1.82) will be an immediate consequence of (1.74) and of the equality $\mathrm{Irr}_z(\mathscr{F}) = -i_z(\mathfrak{F})$ *between the formal and analytic irregularities (cf. Proposition* **??***), which mainly involves radii and not cohomology.*

In Section **??** *and Appendix* **??***, we obtain a stronger result about the comparison between analytic and meromorphic cohomologies, under certain conditions that do not necessarily involves the Liouvilleness of the exponents.*

In particular, a consequence of these comparison results will be the fact that each local analytic solution of $\mathcal{F}$ *with possibly an essential pole at* z *is actually* meromorphic at z *(cf. Remark* **??***).*

1.6. Overconvergence.

In this section, we fix some definitions for the overconvergent setting. Let X be a quasi-smooth K-analytic curve.

Definition 1.6.1 (Overconvergent functions)**.** *For every subset* V *of* X*, we set*

$$\mathscr{O}^\dagger_X(V) \;:=\; \varinjlim_{U\supseteq V} \mathscr{O}(U)\;, \tag{1.83}$$

where U *runs through the family of analytic domains of* X *that are neighborhoods of* V*.*

We often write $\mathscr{O}^\dagger(V) := \mathscr{O}^\dagger_X(V)$ *if no confusion is possible.*

Remark 1.6.2. *The definition is completely satisfactory only if* $V \subseteq \mathrm{Int}(X)$*. In rigid cohomology, this is often fulfilled by embedding* V *into a projective curve* $\overline{X}$ *(for which* $\mathrm{Int}(\overline{X}) = \overline{X}$*) and considering* $\mathscr{O}^\dagger_{\overline{X}}(V)$*.*

Recall that a smooth curve is a quasi-smooth curve with no boundary.

Definition 1.6.3. *A differential equation* $(\mathscr{F},\nabla)$ *on* X *is said to be* partially overconvergent *(resp.* overconvergent*) if there exists a quasi-smooth (resp.* smooth*)* K*-analytic curve* X' *and a differential equation* $(\mathscr{F}',\nabla')$ *on* X' *such that* X *embeds as an analytic domain in* X' *and* $(\mathscr{F}',\nabla')$ *restricts to* $(\mathscr{F},\nabla)$ *on* X*.*

To emphasize overconvergence, we often write

$$(\mathscr{F},\nabla) \;=\; (\mathscr{F}',\nabla')_{|X^\dagger}\;. \tag{1.84}$$

We state the following definition by analogy with rigid cohomology.

Definition 1.6.4 (Overconvergent isocrystals)**.** *Let* V *be an elementary tube centered at a point* $x \in X$ *of type* 2 *or* 3*. An overconvergent isocrystal over* V *is a overconvergent differential equation* $\mathscr{G}$ *over* V *such that*

$$\mathcal{R}_{\{x\},1}(x,\mathscr{G}) \;=\; 1\;. \tag{1.85}$$

Remark 1.6.5. *Let* $\mathscr{G}$ *be an overconvergent isocrystal over an elementary tube* V *centered at* x*. By [PP13, 6.1.4, ii)], all the radii* $\mathcal{R}_{\{x\},i}(-,\mathscr{G})$ *are constant with value 1 on* V*. In particular,* $\mathscr{G}$ *is* *trivial on each virtual open disk in* $V - \{x\}$*.*

As a consequence, if $\mathscr{G}$ *is considered as a differential module on an elementary neighborhood* U *of* V*, then the germs of segments out of* x *that belong to* $\Gamma_{\{x\},i}(\mathscr{G})$ *are exactly those of* $\mathrm{Sing}(x,V)$ *and we have*

$$\Gamma_{\{x\},i}(\mathscr{G}) = \Gamma_U \quad \textit{for all } i \in \{1,\ldots,\mathrm{rank}(\mathscr{G})\}\ , \tag{1.86}$$

where Γ_U *denotes the skeleton of the pseudo-triangulation* $\{x\}$ *of* U*.*

In the following, we fix a partially overconvergent differential equation $(\mathscr{F},\nabla)$ on X: there exists a K-analytic curve X' and a differential equation $(\mathscr{F}',\nabla')$ on X' such that X embeds as an analytic domain in X' and $(\mathscr{F}',\nabla')$ restricts to $(\mathscr{F},\nabla)$ on X.

Definition 1.6.6. *The overconvergent de Rham cohomology of* $(\mathscr{F},\nabla)$ *on* X *is defined as the inductive limit of the de Rham cohomology of* $(\mathscr{F}',\nabla')$ *on the neighborhoods* U *of* X *in* X'*:*

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X^\dagger,\mathscr{F}) \ := \ \varinjlim_U \ \mathrm{H}^\bullet_{\mathrm{dR}}(U,\mathscr{F}'_{|U})\ . \tag{1.87}$$

The previous definitions can be adapted to the meromorphic setting. We use the same notations as in Section 1.5.1.

Definition 1.6.7. *Let* Z *be a locally finite subset of rigid points of* X*. A differential equation* $(\mathcal{F},\nabla)$ *on* X *with meromorphic singularities on* Z *is said to be* partially overconvergent *(resp.* overconvergent*) if* X *may be embedded as an analytic domain in a quasi-smooth (resp.* smooth*)* K*-analytic curve* X' *such that* Z *is locally finite in* X' *and if there exists a differential equation* $(\mathcal{F}',\nabla')$ *on* X' *with meromorphic singularities on* Z *such that* $(\mathcal{F}',\nabla')$ *restricts to* $(\mathcal{F},\nabla)$ *on* X*.*

To emphasize overconvergence, we often write

$$(\mathcal{F},\nabla) \ = \ (\mathcal{F}',\nabla')_{|X^\dagger}\ . \tag{1.88}$$

Set $Y := X - Z$. The restriction of the differential equation $(\mathcal{F},\nabla)$ to Y is a partially overconvergent (resp. overconvergent) differential equation on Y that we often denote by

$$\mathscr{F} \ := \ \mathcal{F}_{|Y^\dagger}\ . \tag{1.89}$$

Definition 1.6.8. *The overconvergent de Rham cohomology of* $(\mathcal{F},\nabla)$ *on* X *is defined as the inductive limit of the de Rham cohomology of* $(\mathcal{F}',\nabla')$ *on the neighborhoods* U *of* X *in* X'*:*

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X^\dagger(*Z),\mathcal{F}) \ := \ \varinjlim_U \ \mathrm{H}^\bullet_{\mathrm{dR}}(U(*Z),\mathcal{F}'_{|U})\ . \tag{1.90}$$

Fix the setting as in Definition 1.6.7. Since $\mathcal{F}'$ has no meromorphic singularities outside X, it makes sense to consider Liouville conditions along the germs of segments of X' that do not belong to X.

Definition 1.6.9. *Let* $x \in \partial X$*. Denote by* $b_{x,1},\ldots,b_{x,t_x}$ *the germs of segments out of* x *that belong to* X' *but not to* X*. We say that* $\mathcal{F}$ *is free (resp. strongly free) of overconvergent Liouville numbers at* x *if, for every* $i \in \{1,\ldots,t_x\}$*,* $\mathscr{F}'$ *is free (resp. strongly free) of Liouville numbers along* $b_{x,i}$ *(cf. Definition* **??***).*

Remark 1.6.10. *Let* $i \in \{1,\ldots,t_x\}$*. By the finiteness of the radii (cf. [Pul15] and [PP15]), there exists a virtual open annulus* C_i *whose skeleton* Γ_{C_i} *(suitably oriented) represents* $b_{x,i}$ *such that the radii of* $\mathcal{F}'$ *and* $\mathrm{End}(\mathcal{F}')$ *are log-affine along* Γ_{C_i}*. In the setting of Definition 1.6.9, Lemma* **??** *ensures that* $\mathcal{F}'$ *is (resp.* $\mathcal{F}'$ *and* $\mathrm{End}(\mathcal{F}')$ *are) free of Liouville numbers along the whole* Γ_{C_i}*.*

Recall that an elementary neighborhood U of X in X' is *adapted to* $\mathcal{F}$ if $U-Z$ is an elementary neighborhood of $X-Z$ in X' that is adapted to the restricted equation $\mathscr{F}'_{|X'-Z}$ (cf. Definition 2.1.17).

Notice that, in the following lemma, no Liouville conditions are required at the germs of segments that lie at the open boundary of X.

Lemma 1.6.11. *Assume that*

i) $\mathcal{F}$ is partially overconvergent on X;

ii) $\mathcal{F}$ is free of overconvergent Liouville numbers at every point of ∂X.

Let U be an elementary neighborhood of X in X' that is adapted to $\mathcal{F}'$. Then, for all $i \geqslant 0$, we have a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(U(*Z),\mathcal{F}') \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X^\dagger(*Z),\mathcal{F}) \;. \tag{1.91}$$

Proof. For each $x \in \partial X$, we denote by $b_{x,1},\ldots,b_{x,t_x}$ the germs of segments out of x that belong to X' but not to X. By definition, $U-X$ is a disjoint union of virtual open annuli $\{C_{x,i}\}_{x\in\partial X, i=1,\ldots,t_x}$ representing the germs of segments $\{b_{x,i}\}_{x\in\partial X,i=1,\ldots,t_x}$. For $x\in\partial X$ and $i\in\{1,\ldots,t_x\}$, we denote by $\Gamma_{x,i}$ the skeleton of $C_{x,i}$. The radii of $\mathcal{F}'$ is log-affine along it, by assumption.

We may write $U = X\cup\bigcup_{x\in\partial X,1\leqslant i\leqslant t_x} C_{x,i}$ as in (1.8). Remark that the pseudo-triangulation S of X is also a pseudo-triangulation of U. The skeleton of the latter is $\Gamma := \Gamma_S\cup\bigcup_{x\in\partial X,1\leqslant i\leqslant t_x}\Gamma_{x,i}$. By assumption, $\mathcal{F}'_{|C_{x,i}}$ is free of Liouville numbers (cf. Remark 1.6.10).

If we replace every annulus $C_{x,i}$ by a smaller annulus $C'_{x,i}$ with the same properties, we find another elementary neighborhood U' adapted to $\mathcal{F}'$ such that the natural maps $\mathrm{H}^i_{\mathrm{dR}}(U,\mathscr{F}'_{|U}) \to \mathrm{H}^i_{\mathrm{dR}}(U',\mathscr{F}'_{|U'})$, for $i=0,1$, are isomorphisms. Indeed, we have $U = U'\cup\bigcup_{x\in\partial X,1\leqslant i\leqslant t_x} C_{x,i}$, $U'\cap\bigcup_{x\in\partial X,1\leqslant i\leqslant t_x} C_{x,i} = \bigcup_{x\in\partial X,1\leqslant i\leqslant t_x} C'_{x,i}$ and, for every $x\in\partial X$ and $i\in\{1,\ldots,t_x\}$, the restriction map from $C_{x,i}$ to $C'_{x,i}$ induces isomorphisms in cohomology by [?, Corollary A.3.2 and Lemma 2.3.6]. We conclude by the Mayer-Vietoris exact sequence.

We have found a cofinal family of neighborhoods of X with isomorphic cohomologies. We deduce that the natural maps $\mathrm{H}^i_{\mathrm{dR}}(U(*Z),\mathcal{F}'_{|U}) \to \mathrm{H}^i_{\mathrm{dR}}(X^\dagger(*Z),\mathcal{F})$ are isomorphisms for all $i\geqslant 0$. □

Lemma 1.6.12. *The following assertions are equivalent:*

i) $\mathcal{F}$ is free (resp. strongly free) of Liouville numbers at the germs of segments at the open boundary of X and free (resp. strongly free) of overconvergent Liouville numbers at each point of ∂X;

ii) there exists an elementary neighborhood U of X in X' adapted to $\mathcal{F}'$ such that $\mathcal{F}'_{|U}$ is free (resp. strongly free) of Liouville numbers at every germ of segment at the open boundary of U;

iii) for every elementary neighborhood U of X in X' adapted to $\mathcal{F}'$, $\mathcal{F}'_{|U}$ is free (resp. strongly free) of Liouville numbers at every germ of segment at the open boundary of U. □

1.7. Some situations with trivial cohomology groups.

We now provide some conditions to ensure that we have trivial cohomology in the spectral non-solvable case.

Proposition 1.7.1. *Let X be a connected quasi-smooth curve, and let $\mathscr{F}$ be a differential equation on X. Assume that*

i) $\Gamma_S\neq\emptyset$ (i.e. X is not a virtual open disk with empty pseudo-triangulation);

ii) $\Gamma_S(\mathscr{F}) = \Gamma_S$;

iii) all the radii of $\mathscr{F}$ are spectral non-solvable at each point of Γ_S.

Then, for all i, we have

$$\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \;=\; 0\;. \tag{1.92}$$

Proof. In [?, Proposition 1.4.9], we have proven that we have $\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F}) = \mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) = 0$ in the following two situations:

a) X is an open pseudo-annulus endowed with the empty pseudo-triangulation and all the radii of $\mathscr{F}$ are log-affine along Γ_X and strictly smaller than 1;

b) X is a virtual open disk and all the radii of $\mathscr{F}$ are constant on X and strictly smaller than 1.

Moreover, using exactly the same arguments as in step 1 of the proof of [?, Proposition 1.4.9], one proves that the result of the proposition holds when X is cohomologically Stein and Ω^1_X is free.

Let us now prove the result of the statement in full generality. For each point x of S, choose an open neighborhood U_x of x such that

i) $\{x\}$ is a pseudo-triangulation of U_x adapted to $\mathscr{F}_{|U_x}$;

ii) $\Gamma_{\{x\}}(\mathscr{F}_{|U_x}) = \Gamma_{\{x\}}$, where $\Gamma_{\{x\}}$ is the skeleton of the pseudo-triangulation $\{x\}$ of U_x;

iii) U_x is cohomologically Stein;

iv) $\Omega^1_{U_x}$ is free;

Up to shrinking them, we may assume that all the U_x's are disjoint. Set $U := \bigcup_{x\in S} U_x$. By the discussion at the beginning of the proof, we have $\mathrm{H}^0_{\mathrm{dR}}(U,\mathscr{F}_{|U}) = \mathrm{H}^1_{\mathrm{dR}}(U,\mathscr{F}_{|U}) = 0$.

Let V be the union of the connected components of $X - S$ that are not contained in U. Since all these connected components are virtual open disks or open pseudo-annuli, we may use [?, Proposition 1.4.9]. We deduce that $\mathrm{H}^0_{\mathrm{dR}}(V,\mathscr{F}_{|V}) = \mathrm{H}^1_{\mathrm{dR}}(V,\mathscr{F}_{|V}) = 0$.

By construction, $U\cap V$ is a disjoint union of open pseudo-annuli and all the radii of $\mathscr{F}$ are locally constant outside their skeletons. By [?, Proposition 1.4.9] again, we have $\mathrm{H}^0_{\mathrm{dR}}(U\cap V,\mathscr{F}_{|U\cap V}) = \mathrm{H}^1_{\mathrm{dR}}(U\cap V,\mathscr{F}_{|U\cap V}) = 0$.

The result now follows from the Mayer-Vietoris long exact sequence.

□

In the following, we use the notions of elementary tube and elementary neighborhood (see Definitions 1.3.1 and 1.3.4 respectively).

Corollary 1.7.2. *Let $x \in X$ be a point of type 2 or 3. Let V be an elementary tube centered at x that is adapted to $\mathscr{F}$. Assume that all the radii of $\mathscr{F}$ are spectral and non-solvable at x. Then, for every elementary neighborhood U of V in X that is adapted to $\mathscr{F}$ and for every $i \geqslant 0$, we have*

$$\mathrm{H}^i_{\mathrm{dR}}(U,\mathscr{F}) \;=\; 0\;. \tag{1.93}$$

In particular, for the overconvergent cohomology on V with respect to X, for all $i \geqslant 0$, we have

$$\mathrm{H}^i_{\mathrm{dR}}(V^\dagger,\mathscr{F}) \;=\; 0\;. \tag{1.94}$$

Proof. Let U be an elementary neighborhood of V in X that is adapted to $\mathscr{F}$. Endow it with the triangulation $\{x\}$ (see Remark 1.3.5). It follows form the hypotheses that we have

i) $\Gamma_{\{x\}}(\mathscr{F}) = \Gamma_{\{x\}}$;

ii) all the radii of $\mathscr{F}_{|U}$ are spectral and non-solvable at every point of $\Gamma_{\{x\}}$.

We conclude by Proposition 1.7.1. □

The following corollary is often useful to remove the boundary.

Corollary 1.7.3. *Let Z be a locally finite subset of rigid points of X. Let $\mathcal{F}$ be a differential equation on X with meromorphic singularities on Z and assume that all its radii are spectral non-solvable on ∂X. For every $x \in \partial X$, choose an elementary tube V_x in $X - Z$ centered at x that is adapted to $\mathcal{F}_{|X-Z}$ (cf. Definition 1.3.1). Set*

$$Y \;:=\; X - \bigcup_{x\in\partial X} V_x\;. \tag{1.95}$$

Then, for $i = 0, 1$, we have canonical isomorphisms

$$\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(\mathrm{Int}(X)(*Z),\mathcal{F}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(Y(*Z),\mathcal{F})\;. \tag{1.96}$$

Proof. Let $x \in \partial X$ and let U_x be an elementary neighborhood of V_x in $X - Z$ that is adapted to $\mathcal{F}_{|X-Z}$ (cf. Definition 1.3.4). By Remark 1.5.3 and Corollary 1.7.2, for every $i \geqslant 0$, we have $\mathrm{H}^i_{\mathrm{dR}}(U_x(*Z),\mathcal{F}_{|U_x}) = \mathrm{H}^i_{\mathrm{dR}}(U_x,\mathcal{F}_{|U_x}) = 0$. Up to shrinking the U_x's, we may assume that they are all disjoint. Set $U := \bigcup_{x\in\partial X} U_x$. Then, for every $i \geqslant 0$, we have $\mathrm{H}^i_{\mathrm{dR}}(U(*Z),\mathcal{F}_{|U}) = \mathrm{H}^i_{\mathrm{dR}}(U,\mathcal{F}_{|U}) = 0$.

The intersection $U \cap Y$ is a disjoint union of virtual open annuli on which the radii are all spectral non-solvable and log-affine, hence, by Proposition 1.7.1, we have $\mathrm{H}^i_{\mathrm{dR}}((U\cap Y)(*Z),\mathcal{F}_{|U\cap Y}) = \mathrm{H}^i_{\mathrm{dR}}(U\cap Y,\mathcal{F}_{|U\cap Y}) = 0$ for all $i \geqslant 0$.

Since $\{Y, U\}$ is a covering of X, the isomorphism $\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(Y(*Z),\mathcal{F}_{|Y})$ now follows from the Mayer-Vietoris exact sequence (see Lemma 1.4.3).

Finally, $\mathrm{Int}(X) = X - \partial X$ is the disjoint union of Y and of a family of disjoint virtual open disks not meeting Z on which the radii are all constant. For every such disk D, the non-solvability assumption implies that $\mathrm{H}^0_{\mathrm{dR}}(D(*Z),\mathcal{F}_{|D}) = \mathrm{H}^0_{\mathrm{dR}}(D,\mathcal{F}_{|D}) = 0$ and it follows from Corollary **??** that we also have $\mathrm{H}^1_{\mathrm{dR}}(D(*Z),\mathcal{F}_{|D}) = \mathrm{H}^1_{\mathrm{dR}}(D,\mathcal{F}_{|D}) = 0$. The result now follows from the Mayer-Vietoris exact sequence as before. □

Remark 1.7.4. *Notice that if the curve X is finite, then so is the curve Y (cf. (1.95)). This follows from the finiteness of ∂X and the local finiteness of $\Gamma_S(\mathscr{F})$ (cf. Theorem* **??***).*

2. Local and global irregularities and local virtual indexes

Let X be a quasi-smooth K-analytic curve and let $\mathscr{F}$ be a differential equation on X.

In this section, we introduce the notion of global irregularity of the differential equation $\mathscr{F}$ and investigate some of its properties. The importance of this notion relies on the fact that, under appropriate conditions, is controls the finite-dimensionality of the de Rham cohomology groups (see Section 4).

In [**?**, Section 1.2], we have defined a notion of local irregularity. For the convenience of the reader, we state it here again. Recall that a germ of segment b in X is said to be good if it may be represented by the skeleton of an open pseudo-annulus contained in X. For such a germ b, we say that $\mathscr{F}$ has log-affine total height along b if there exists an open pseudo-annulus C in X whose skeleton (suitably oriented) represents b such that the total height function $H_{\emptyset,r}(-,\mathscr{F}_{|C})$, where r be the rank of $\mathscr{F}$ around b, is log-affine on Γ_C (where C is endowed with the trivial pseudo-triangulation).

Definition 2.0.1 (Irregularity over a good germ of segment)**.** *Let b be a good germ of segment in X on which $\mathscr{F}$ has log-affine total height. Let C be an open pseudo-annulus whose skeleton Γ_C (suitably oriented) represents b and such that the total height function $H_{\emptyset,r}(-,\mathscr{F}_{|C})$ is log-affine on Γ_C, where*

$r = \mathrm{rank}(\mathscr{F}_{|C})$. *We define the irregularity of* $\mathscr{F}$ *along* b *as*

$$\mathrm{Irr}_b(\mathscr{F}) \;:=\; -\deg(b)\cdot\partial_b H_{\emptyset,r}(-,\mathscr{F}_{|C}) \;\in\; \mathbb{Z}\;. \tag{2.1}$$

2.1. Global irregularity

In this section, we fix a pseudo-triangulation S of X. We first introduce here a super-harmonicity condition.

If $\Gamma \subseteq X$ is a graph, and if $x \in \Gamma$, we denote by $\mathscr{B}(x,\Gamma)$ the set of germs of segments out of x that belong to Γ. If Γ' is another graph containing x we denote by

$$\mathscr{B}(x,\Gamma-\Gamma') \tag{2.2}$$

the set of germs of segment out of x that belong to Γ but not to Γ' (cf. [PP13, Notation 1.1.41]). We also set (cf. [PP13, Notation 1.1.51])

$$\chi(x,\Gamma) \;:=\; 2\deg(x)-2g(x)-\sum_{b\in\mathscr{B}(x,\Gamma)}\deg(b)\;. \tag{2.3}$$

If $\Gamma=\Gamma_S$ we set

$$\chi(x,S) \;:=\; \chi(x,\Gamma_S)\;. \tag{2.4}$$

Recall the definition of Laplacian dd^c (cf. [PP13, Definition 1.1.22]), the definition of i_x^{sp} (cf. [PP13, Definition 2.6.2]) and the notion of vertex free of solvability (cf. [PP13, Definition 2.6.1]).

Theorem 2.1.1 (Weak super-harmonicity, [Ked15b, Theorem 5.3.6], [PP13, Theorem 6.2.27], [BP18, Corollary 4.11])**.** *Let x be a point of type 2 or 3 in $\Gamma_S\cap\mathrm{Int}(X)$. Then, we have*

$$dd^c H_{S,i}(x,\mathscr{F}) \;\leqslant\; -\chi(x,S)\cdot\min(i,i_x^{\mathrm{sp}}) \tag{2.5}$$

with equality whenever i is a vertex free of solvability at x. □

Definition 2.1.2. *Let $x\in X$. For all $i\in\{1,\dots,i_x^{\mathrm{sp}}\}$, we call i-*th intrinsic Laplacian of $\mathscr{F}$ at x *the number*

$$\Delta_i(x,\mathscr{F},S) \;:=\; \begin{cases} dd^c H_{S,i}(x,\mathscr{F}) & \text{if} \quad x\notin\Gamma_S\,,\\ dd^c H_{S,i}(x,\mathscr{F})+i\cdot\chi(x,S) & \text{if} \quad x\in\Gamma_S\,.\end{cases} \tag{2.6}$$

We simply write $\Delta_i(x,\mathscr{F})$ if no confusion is possible.

Remark 2.1.3. *i) For $x\in\Gamma_S-S$, we have $\chi(x,S)=0$, hence the two lines of (2.6) give the same result.*

ii) If X is an affinoid domain of $\mathbb{A}_K^{1,\mathrm{an}}$, then $\Delta_i(x,\mathscr{F})$ coincides with the Laplacian of the non-intrinsic (or spectral) partial height $H_i^{\mathscr{F}}(x)$ defined in [Pul15].

Lemma 2.1.4. *Let $x\in X$. For all $i\in\{1,\dots,i_x^{\mathrm{sp}}\}$, $\Delta_i(x,\mathscr{F})$ is independent of S.*

Proof. We can assume K algebraically closed. Let S' be a pseudo-triangulation of X. There exists a pseudo-triangulation containing S and S', so without loss of generality we assume that $S\subseteq S'$.

Let $i\in\{1,\dots,i_x^{\mathrm{sp}}\}$. Let us first assume that $x\notin\Gamma_{S'}$. Then, we have $dd^cH_{S,i}(x,\mathscr{F})=dd^cH_{S',i}(x,\mathscr{F})$, because the slopes of the radii are all unchanged by localization to the connected component D of $X-\Gamma_{S'}$ containing x.

Let us now assume that $x\in\Gamma_{S'}-\Gamma_S$. Denote by $b_{x,\infty}$ the germ of segment out of x directed towards Γ_S. Then, for every $j\in\{1,\dots,i_x^{\mathrm{sp}}\}$ and every germ of segment b out of x, we have (cf.

[PP13, Proposition 2.7.1])

$$\partial_b \mathcal{R}_{S',j}(x,\mathscr{F}) \;=\; \begin{cases} \partial_b \mathcal{R}_{S,j}(x,\mathscr{F}) & \text{if} \quad b \notin \mathscr{B}(x,\Gamma_{S'}) \\ \partial_b \mathcal{R}_{S,j}(x,\mathscr{F}) - 1 & \text{if} \quad b = b_{x,\infty} \\ \partial_b \mathcal{R}_{S,j}(x,\mathscr{F}) + 1 & \text{if} \quad b \in \mathscr{B}(x,\Gamma_{S'}) - \{b_{x,\infty}\}\,. \end{cases} \tag{2.7}$$

We deduce that $dd^c H_{S',i}(x,\mathscr{F}) = dd^c H_{S,i}(x,\mathscr{F}) - i + i(N_{S'}(x)-1)$, by noting that $N_{S'}(x)-1$ is the cardinality of $\mathscr{B}(x,\Gamma_{S'}) - \{b_{x,\infty}\}$. Since $x \notin \Gamma_S$, we have $g(x) = 0$, so $\chi(x,S') = 2 - N_{S'}(x)$, and we conclude.

The case where $x \in \Gamma_S$ is similar. □

Corollary 2.1.5. *Let Y be an open subset of X. For every $x \in Y$ and $i \in \{1,\ldots,i_x^{\mathrm{sp}}\}$, we have*

$$\Delta_i(x,\mathscr{F}) \;=\; \Delta_i(x,\mathscr{F}_{|Y})\,. \tag{2.8}$$

Proof. Let $x \in Y$ and $i \in \{1,\ldots,i_x^{\mathrm{sp}}\}$. We can find an open neighborhood U of x in Y and a pseudo-triangulation S' of U that extends both to a pseudo-triangulation of Y and a pseudo-triangulation of X. In this case, we have

$$\Delta_i(x,\mathscr{F}) \;=\; \Delta_i(x,\mathscr{F}_{|U}) \;=\; \Delta_i(x,\mathscr{F}_{|Y})\,. \tag{2.9}$$

□

We now introduce a general notion of global irregularity of a differential equation $\mathscr{F}$ over a K-analytic curve X that is natural and may be useful for future applications.

For a point x in X and a germ of segment b out of x, if the connected component of $X - \{x\}$ containing b is a virtual open disk that does not meet Γ_S, we will denote it by D_b (see [PP13, Notations 2.8.3] for more details).

Recall also that all germs of segments at the open boundary $\partial^o X$ (resp. out of a point $x \in \partial X$) are oriented towards the interior of X (resp. out of x).

Definition 2.1.6. *Let $x \in X$ be a point of type 2 or 3 and let Γ be a subgraph of X that is finite around $x \in \Gamma$. Set $r := \mathrm{rank}(\mathscr{F}_x)$ and*

$$\chi^o(x,\Gamma,\mathscr{F}) \;:=\; r\cdot\chi(x,\Gamma) - \sum_{b\in\mathscr{B}(x,\Gamma)} \mathrm{Irr}_b(\mathscr{F}) \;\in\; \mathbb{Z}\,. \tag{2.10}$$

The following lemma establishes a link between $\chi^o(x,\Gamma_S(\mathscr{F}),\mathscr{F})$ and $\Delta_r(x,\mathscr{F})$.

Lemma 2.1.7. *Let $x \in \Gamma_S$. Set $r := \mathrm{rank}(\mathscr{F}_x)$. Then, we have*

$$\Delta_r(x,\mathscr{F}) \;=\; r\cdot\chi(x,S) + dd^c H_{S,r}(x,\mathscr{F})\,, \tag{2.11}$$

$$\;=\; r\cdot\chi(x,\Gamma_S(\mathscr{F})) - \sum_{b\in\mathscr{B}(x,\Gamma_S(\mathscr{F}))} \mathrm{Irr}_b\,\mathscr{F} + \sum_{b\in\mathscr{B}(x,\Gamma_S(\mathscr{F})-\Gamma_S)} h^0(D_b,\mathscr{F})\,. \tag{2.12}$$

In particular, if the radii of $\mathscr{F}$ are all spectral non-solvable at x, we have

$$\Delta_r(x,\mathscr{F}) \;=\; r\cdot\chi(x,\Gamma_S(\mathscr{F})) - \sum_{b\in\mathscr{B}(x,\Gamma_S(\mathscr{F}))} \mathrm{Irr}_b\,\mathscr{F} \tag{2.13}$$

$$\;=\; \chi^o(x,\Gamma_S(\mathscr{F}),\mathscr{F})\,. \tag{2.14}$$

Proof. Equality (2.11) is the definition of $\Delta_r(x,\mathscr{F})$ (cf. Definition 2.1.2), while Equality (2.12) follows from [PP13, Lemma 2.8.4] and from $dd^c H_{S,r}(x,\mathscr{F}) = \sum_{b\in\mathscr{B}(x,\Gamma_S(\mathscr{F}))} \partial_b H_{S,r}(x,\mathscr{F})$. Equation (2.14) is a direct consequence. □

We now provide the principal definition of this section. Recall that a curve X is *finite* if it has finitely many connected components and if it admits a finite pseudo-triangulation. (cf. Definition **??**).

Definition 2.1.8 (Global Irregularity)**.** *Assume that X is a finite curve. We say that $\mathscr{F}$ has* well-defined irregularity *if, for every germ of segment $b \in \partial^o X$, the total height $H_{S,r_b}(-,\mathscr{F})$ is log-affine along b (cf. Definition **??**), where r_b is the local rank of $\mathscr{F}$ at b.*

In this case, we define the global irregularity *of $\mathscr{F}$ as*[3]

$$\mathrm{Irr}_X(\mathscr{F}) \;:=\; \sum_{x\in\partial X}\Delta_{r_x}(x,\mathscr{F}) - \sum_{b\in\partial^o X}\mathrm{Irr}_b(\mathscr{F}) \;\in\; \mathbb{Z}\,. \tag{2.15}$$

Remark 2.1.9. *If K is trivially valued, it follows from Remark **??** that $\mathscr{F}$ always has well-defined irregularity.*

Lemma 2.1.10. *Assume that the radii of $\mathscr{F}$ are all spectral non-solvable at each point of ∂X. Then, the definition of the global irregularity $\mathrm{Irr}_X(\mathscr{F})$ is independent of the choice of S.*

Proof. Let $x \in \partial X$ and set $r_x := \mathrm{rank}(\mathscr{F}_x)$. By Lemma 2.1.4, $\Delta_{r_x}(x,\mathscr{F})$ is independent of S. Moreover, Definition 2.0.1 is also independent of S □

Remark 2.1.11. *Notice that there are equations with well-defined irregularity whose radii are not log-affine at the open boundary (see example 4.1.6).*

Proposition 2.1.12. *Let U and V be open subsets of a quasi-smooth K-analytic curve X and let $\mathscr{F}$ be a differential equation on X. Assume that U and V are finite curves and that $\mathscr{F}$ (suitably restricted) has well-defined irregularity on U and V. Assume moreover that, at each $x \in \partial U \cup \partial V$, the radii of $\mathscr{F}$ are all spectral non-solvable.*

Then $U \cup V$ and $U \cap V$ are finite curves too, $\mathscr{F}$ has well-defined irregularity on them and we have

$$\mathrm{Irr}_{U\cup V}(\mathscr{F}_{|U\cup V}) \;=\; \mathrm{Irr}_U(\mathscr{F}_{|U}) + \mathrm{Irr}_V(\mathscr{F}_{|V}) - \mathrm{Irr}_{U\cap V}(\mathscr{F}_{|U\cap V}). \tag{2.16}$$

Proof. We may assume K algebraically closed. Moreover, (2.16) only depends on the restriction of $\mathscr{F}$ to $U \cup V$, therefore we can also assume $X = U \cup V$.

The fact that $U \cup V$ and $U \cap V$ are finite curves follows from [PP13, Lemma 1.1.59].

Since U and V are open, $\partial U \cup \partial V = \partial(U \cup V)$ and $\partial U \cap \partial V = \partial(U \cap V)$.

By Corollary 2.1.5, for every point x in every open subset W of X, we have

$$\Delta_{r_x}(x,\mathscr{F}) \;=\; \Delta_{r_x}(x,\mathscr{F}_{|W})\;, \tag{2.17}$$

where $r_x := \mathrm{rank}(\mathscr{F}_x)$.

For every open subset W of X, denote by $\partial^o_{rc}W$ (resp. $\partial^o_{nrc}W$) the set of germs of segments in the open boundary of W that are (resp. are not) relatively compact in $X = U \cup V$. Then, we have

$$\partial^o_{rc}U \cap \partial^o_{rc}V \;=\; \emptyset \qquad \text{and} \qquad \partial^o_{rc}U \cup \partial^o_{rc}V \;=\; \partial^o_{rc}(U\cap V)\;, \tag{2.18}$$

$$\partial^o_{nrc}U \cup \partial^o_{nrc}V \;=\; \partial^o X \qquad \text{and} \qquad \partial^o_{nrc}U \cap \partial^o_{nrc}V \;=\; \partial^o_{nrc}(U\cap V)\;. \tag{2.19}$$

The claim follows. □

Proposition 2.1.13. *Assume that the curve X is finite. Let*

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0 \tag{2.20}$$

[3] If ∂X or $\partial^o X$ is empty, the corresponding sum evaluates to 0.

be an exact sequence of differential equations on X. Then, the following properties hold.

i) Let $x \in \partial X$ be a point of the boundary. If $\mathscr{F}_1$ and $\mathscr{F}_3$ both have spectral non-solvable radii at x, then so has $\mathscr{F}_2$. In this case, one has

$$\Delta_{r_2}(x, \mathscr{F}_2) \;=\; \Delta_{r_1}(x, \mathscr{F}_1) + \Delta_{r_3}(x, \mathscr{F}_3)\;, \tag{2.21}$$

where r_i is the local rank of $\mathscr{F}_i$ at x.

ii) $\mathscr{F}_2$ has well-defined irregularity if, and only if, so have both $\mathscr{F}_1$ and $\mathscr{F}_3$.

iii) Assume that $\mathscr{F}_1$ and $\mathscr{F}_3$ both have spectral non-solvable radii at each point $x \in \partial X$ of the boundary. Then, if $\mathscr{F}_1$ and $\mathscr{F}_3$ have well-defined irregularities on X, then so has $\mathscr{F}_2$ and, in this case, we have

$$\mathrm{Irr}_X(\mathscr{F}_2) \;=\; \mathrm{Irr}_X(\mathscr{F}_1) + \mathrm{Irr}_X(\mathscr{F}_3)\;. \tag{2.22}$$

Proof. i) Since $\mathscr{F}_1$ and $\mathscr{F}_3$ both have spectral non-solvable radii at x, if follows from the spectral definition of the radii (cf. [Ked10, Definition 9.8.1] or [Pul15, Section 4.2]) that the radii of $\mathscr{F}_2$ at x are the union with multiplicity of those of $\mathscr{F}_1$ and $\mathscr{F}_3$ at x. So $\mathscr{F}_2$ also has spectral non-solvable radii along each germ of segment out of x. Spectral radii behave well with respect to exact sequences (cf. [PP13, Proposition 3.6.1]), so (2.21) holds.

ii) We only have to check the log-affinity of the total heights at the open boundary of X, which follows from Proposition **??**.

iii) It is a consequence of Proposition **??** for the germs of segment at infinity and of point i) for the boundary. □

The following lemma is naturally related to Corollary 1.7.3.

Proposition 2.1.14. *Assume that X is a connected finite curve. Set $r := \mathrm{rank}(\mathscr{F}_x)$. For every $x \in \partial X$, choose an elementary tube V_x centered at x that is adapted to $\mathscr{F}$ (cf. Definition 1.3.1). Set*

$$Y \;:=\; X - \bigcup_{x\in\partial X} V_x\;. \tag{2.23}$$

Assume that $\mathscr{F}$ has spectral non-solvable radii at each point of the boundary ∂X of X.

Then, $\mathscr{F}$ has well-defined irregularity on X if and only if so has its restriction $\mathscr{F}_{|Y}$ on Y. In this case we have

$$r\cdot\chi_c(X) - \mathrm{Irr}_X(\mathscr{F}) \;=\; r\cdot\chi_c(Y) - \mathrm{Irr}_Y(\mathscr{F}_{|Y})\;. \tag{2.24}$$

Proof. Firstly, note that the curve Y is finite. Let B be the family of germ of segments in X out of a point of ∂X that are contained in Y (i.e. not included in $\bigcup_{x\in\partial X} V_x$). We have

$$\partial^o Y \;=\; \partial^o X \cup B\;. \tag{2.25}$$

Since the controlling graphs of $\mathscr{F}$ on X are locally finite, the radii of $\mathscr{F}$ are log-affine on each relatively compact germ of segment in X, in particular, on those in B. It follows that $\mathscr{F}$ has well-defined irregularity on X if, and only if, $\mathscr{F}_{|Y}$ has well-defined irregularity on Y.

Let us now prove (2.24). For every $x \in \partial X$, let U_x be an elementary neighborhood of V_x that is adapted to $\mathscr{F}$ (cf. Definition 1.3.4). We may assume that $U_x \cap U_y = \emptyset$ for all $x \neq y$. Consider the open covering $X = U \cup Y$, where $U := \bigcup_{x\in\partial X} U_x$.

Since $U\cap Y$ is a finite union of open pseudo-annuli on which the radii[4] of $\mathscr{F}$ are log-affine, $\mathscr{F}_{|U\cap Y}$ has well-defined irregularity and $\mathrm{Irr}_{Y\cap U}(\mathscr{F}_{|Y\cap V}) = 0$, where $Y \cap V$ is endowed with the empty

[4]According to Definition 2.0.1, the radii are considered with respect to the empty pseudo-triangulation (cf. Remark **??**).

pseudo-triangulation. By Proposition 2.1.12, we then have $\mathrm{Irr}_X(\mathscr{F}) = \mathrm{Irr}_Y(\mathscr{F}_{|Y}) + \mathrm{Irr}_U(\mathscr{F}_{|U})$.

For each $x \in \partial X$, endow U_x with the pseudo-triangulation $S_x := \{x\}$. We have

$$\begin{aligned} \mathrm{Irr}_U(\mathscr{F}_{|U}) &= \sum_{x\in\partial X} \mathrm{Irr}_{U_x}(\mathscr{F}_{|U_x}) && (2.26)\\ &= \sum_{x\in\partial X} \Big(\Delta_{r_x}(x,\mathscr{F}_{|U_x}) - \sum_{b\in\partial^o U_x} \mathrm{Irr}_b(\mathscr{F}_{|U_x})\Big) && (2.27)\\ &= \sum_{x\in\partial X} \chi(x,\Gamma_{S_x}(\mathscr{F}_{|U_x}))\cdot r, && (2.28)\end{aligned}$$

by Lemma 2.1.7. By construction, we have $\Gamma_{S_x}(\mathscr{F}_{|U_x}) = \Gamma_{S_x}$ and $\chi(x,\Gamma_{S_x}) = \chi(x,S_x) = \chi_c(U_x)$ and we deduce that

$$\mathrm{Irr}_U(\mathscr{F}_{|U}) = \sum_{x\in\partial X} \chi_c(U_x)\cdot r = \chi_c(U)\cdot r. \tag{2.29}$$

The result now follows from the equality $\chi_c(X) = \chi_c(U) + \chi_c(Y) - \chi_c(U\cap Y)$ (cf. [PP13, Corollary 1.1.55]). and the fact that $\chi_c(U\cap Y) = 0$. □

2.1.1. Irregularity of meromorphic differential equations. Let X be a finite curve, $Z \subset X$ be a finite set of rigid points and $\mathcal{F}$ a differential equation on X with meromorphic singularities on Z. We set $Y := X - Z$ and $\mathscr{F} := \mathcal{F}_{|Y}$. Of course, Y is a finite curve with the same boundary as X.

We anticipate here a result of the next sections (cf. Lemma **??**) that asserts that for all $z \in Z$ the radii of $\mathcal{F}$ are automatically log-affine along the germ of segments b_z out of z. We also use the notation $\mathrm{Irr}_z(\mathcal{F}) := \mathrm{Irr}_{b_z}(\mathscr{F})$ (cf. Definition **??**).

Definition 2.1.15. *We say that $\mathcal{F}$ has* well-defined irregularity *if, for each $b \in \partial^o X$, the total height $H_{S,r_b}(\mathcal{F},-)$ is* log-affine *along b, where r_b is the local rank of $\mathcal{F}$ along b. In this case we set*

$$\mathrm{Irr}_X(\mathcal{F}) := \sum_{x\in\partial X} \Delta_{r_x}(x,\mathscr{F}) - \sum_{b\in\partial^o X} \mathrm{Irr}_b(\mathscr{F}) - \sum_{z\in Z} \mathrm{Irr}_z(\mathscr{F}) \in \mathbb{Z}\,. \tag{2.30}$$

By definition, the irregularity of $\mathcal{F}$ on X is then that of $\mathscr{F}$ on Y:

$$\mathrm{Irr}_X(\mathcal{F}) = \mathrm{Irr}_Y(\mathscr{F})\,. \tag{2.31}$$

Therefore, Definition 2.1.15 (as well as Definition 2.1.8) is independent of the pseudo-triangulation as soon as the radii of $\mathcal{F}$ are spectral non solvable at every point of the boundary $\partial X = \partial Y$. Moreover, Propositions 2.1.12, 2.1.13 and 2.1.14 admit straightforward generalizations to the meromorphic case that we left to the reader.

2.1.2. Overconvergent global irregularity. We now place ourself in the overconvergent setting. Let X be a quasi-smooth K-analytic curve, Z a locally finite set of rigid points of X and $(\mathcal{F},\nabla)$ an overconvergent differential equation on X with meromorphic singularities at Z (cf. Definition 1.6.7): there exists a smooth K-analytic curve (with no boundary) X' such that X embeds as an analytic domain of X', Z is locally finite in X' and there exists a connexion $(\mathcal{F}',\nabla')$ on X' with meromorphic singularities at Z such that $(\mathcal{F}',\nabla')$ restricts to $(\mathcal{F},\nabla)$ on X. We fix this setting in the following. We will abuse notation and not mention explicitly the dependence in $\mathcal{F}'$ and ∇'.

Definition 2.1.16. *Let $x \in \partial X$. Denote by $b_{x,1},\dots,b_{x,t_x}$ the germs of segments out of x (oriented* *away from x) that belong to X' but not to X. Set*

$$\chi_{st}^{\dagger}(x,\mathcal{F}) \;:=\; -\sum_{i=1}^{t_x} \mathrm{Irr}_{b_{x,i}}(\mathcal{F}')\;. \tag{2.32}$$

We will need to extend the notion of elementary neighborhood adapted to $\mathcal{F}$ in the meromorphic context.

Definition 2.1.17. *Let U be an elementary neighborhood of X in X'. We say that U is* adapted *to $\mathcal{F}'$ if $U-Z$ is an elementary neighborhood of $X-Z$ in X' that is adapted to the restricted equation $\mathscr{F}'_{|X'-Z}$ (cf. Definition 1.3.4).*

Lemma 2.1.18. *Assume that X is a finite curve. Let U be an elementary neighborhood of X in X' that is adapted to $\mathcal{F}'$ (cf. Definition 2.1.17). Then, U is a finite curve and $\mathcal{F}'_{|U}$ has well-defined irregularity if, and only if, so has $\mathcal{F}$ viewed as a (non-overconvergent) differential equation over X (cf. Definition 2.1.15). In this case, we have*

$$\mathrm{Irr}_U(\mathcal{F}'_{|U}) = -\sum_{x\in\partial X}\chi_{st}^{\dagger}(x,\mathcal{F}) - \sum_{b\in\partial^o X}\mathrm{Irr}_b(\mathcal{F}) - \sum_{z\in Z}\mathrm{Irr}_z(\mathcal{F})\;. \tag{2.33}$$

It particular, $\mathrm{Irr}_U(\mathcal{F}'_{|U})$ does not depend on U nor on the choice of the pseudo-triangulation.

Proof. By assumption the boundary ∂X of X is finite, so passing from X to U adds only finitely many segments at infinity. It follows that U is still a finite curve.

Since U has no boundary, Lemma 2.1.10 applies and the definition is independent of the choice of S.

Since U is adapted to $\mathscr{F}'$, the total height of $\mathscr{F}'$ is linear on every segment at infinity in U that does not belong to X. It follows that $\mathscr{F}'_{|U}$ has well-defined irregularity if, and only if, $\mathscr{F}$ has.

To prove formula (2.33), it is enough to note that $\partial^o X \subseteq \partial^o U$ and that every element of $\partial^o U - \partial^o X$ corresponds unambiguously to one of the $b_{x,i}$'s of Definition 2.1.16. □

Thanks to the above lemma we are allowed to give the following definition.

Definition 2.1.19. *We say that $\mathcal{F}$ has well-defined irregularity over $X^\dagger$ if its restriction to X has.*

Definition 2.1.20 (Overconvergent global irregularity)**.** *Assume that X is finite and that $\mathcal{F}$ has well-defined irregularity. We define the overconvergent global irregularity of $\mathcal{F}$ to be*

$$\mathrm{Irr}_{X^\dagger}(\mathcal{F}) \;:=\; -\sum_{x\in\partial X}\chi_{st}^{\dagger}(x,\mathcal{F}) - \sum_{b\in\partial^o X}\mathrm{Irr}_b(\mathcal{F}) - \sum_{z\in Z}\mathrm{Irr}_z(\mathcal{F})\;. \tag{2.34}$$

2.2. Interpretation of the global irregularity by means of virtual local indexes

In this section, we express the irregularity as a sum of certain local contributions $\chi(x,\Gamma_S,\mathscr{F})$ that play the role of local indexes (see Proposition 2.2.8). Under appropriate Liouville conditions, $\chi(x,\Gamma_S,\mathscr{F})$ is actually the index of $\mathscr{F}$ over a certain neighborhood of x (cf. Remark 2.2.10). However, in general it merely represents a numerical invariant of the equation. We call it *virtual local index*. This interpretation of the irregularity will be one of the main ingredients to state a necessary and sufficient condition to have finite-dimensionality of de Rham cohomology in the case where the curve is not finite.

In this section, we fix a quasi-smooth K-analytic curve X, a differential equation $\mathscr{F}$ on it and pseudo-triangulation S on X.

2.2.1. Virtual local indexes. Recall that in Definition 2.1.6 we have set

$$\chi^o(x,\Gamma,\mathscr{F}) \;:=\; r\cdot\chi(x,\Gamma) - \sum_{b\in\mathscr{B}(x,\Gamma)} \mathrm{Irr}_b(\mathscr{F}) \;\in\; \mathbb{Z}\,, \tag{2.35}$$

where $\chi(x,\Gamma)$ and $\mathscr{B}(x,\Gamma)$ were defined in (2.3) and $r=\mathrm{rank}(\mathscr{F})$.

The following definition is related to Lemma 2.1.7.

Definition 2.2.1 (Virtual local indexes)**.** *Let $x\in S$. Set $r:=\mathrm{rank}(\mathscr{F}_x)$. We define the local virtual index $\chi(x,\Gamma_S,\mathscr{F})$ of $\mathscr{F}$ at x as follows:*

$$\chi(x,\Gamma_S,\mathscr{F}) \;=\; \begin{cases} \chi^o(x,\Gamma_S,\mathscr{F}) & \text{if} \quad x\in S-\partial X\;;\\ \chi^o(x,\Gamma_S,\mathscr{F}) - \Delta_r(x,\mathscr{F}) & \text{if} \quad x\in\partial X\;. \end{cases} \tag{2.36}$$

We will prove that the virtual local indexes $\chi(x,\Gamma_S,\mathscr{F})$ control the global irregularity of the differential equation (see Proposition 2.2.8).

We here provide two useful lemmas. Recall that $\mathscr{B}(x,\Gamma-\Gamma')$ was defined in (2.2).

Lemma 2.2.2. *Let $x\in S$. Set $r:=\mathrm{rank}(\mathscr{F}_x)$.*

i) If $x\in S-\partial X$, then

$$\chi(x,\Gamma_S,\mathscr{F}) \;=\; r\cdot\chi(x,S) + dd^cH_{S,r}(x,\mathscr{F}) - \sum_{b\in\mathscr{B}(\Gamma_S(\mathscr{F})-\Gamma_S)} \deg(b)\cdot\partial_b H_{S,r}(x,\mathscr{F})\,. \tag{2.37}$$

ii) If $x\in\partial X$ and if the radii of $\mathscr{F}$ are all spectral non-solvable at x, then

$$\chi(x,\Gamma_S,\mathscr{F}) \;=\; -\sum_{b\in\mathscr{B}(\Gamma_S(\mathscr{F})-\Gamma_S)} \deg(b)\cdot\partial_b H_{S,r}(x,\mathscr{F}) \tag{2.38}$$

Proof. Item i) follows from [PP13, Lemma 2.8.4, item i)].

ii) By (2.14) and Definition 2.1.2, we have

$$\chi^o(x,\Gamma_S(\mathscr{F}),\mathscr{F}) \;=\; \Delta_r(x,\mathscr{F}) \;=\; r\cdot\chi(x,S) + dd^cH_{S,r}(x,\mathscr{F})\,. \tag{2.39}$$

The claim then follows from [PP13, Lemma 2.8.4, item ii)] and Definitions 2.0.1 and 2.1.6. □

Lemma 2.2.3. *Let $x\in S$ and set $r:=\mathrm{rank}(\mathscr{F}_x)$. Then,*

$$\begin{aligned}\chi^o(x,\Gamma_S,\mathscr{F}) \;=\; & \underbrace{(r-i_x^{\mathrm{sp}})\cdot\chi(x,S)}_{A} + \underbrace{\sum_{b\in\mathscr{B}(x,\Gamma_S(\mathscr{F})-\Gamma_S)} -\deg(b)\cdot\partial_b H_{S,i_x^{\mathrm{sp}}}(x,\mathscr{F})}_{B} \\ & + \underbrace{\sum_{b\in\mathscr{B}(x,\Gamma_S)} \deg(b)\cdot\sum_{j=i_x^{\mathrm{sp}}+1}^{r} \partial_b\mathcal{R}_{S,j}(x,\mathscr{F})}_{C}\,. \end{aligned} \tag{2.40}$$

In this case we have

i) $B\leqslant 0$ and $C\leqslant 0$;

ii) $B=0$ if, and only if, for every $b\in\mathscr{B}(x,\Gamma_S(\mathscr{F})-\Gamma_S)$, we have $\partial_b H_{S,i_x^{\mathrm{sp}}}(-,\mathscr{F})=0$;

iii) $C=0$ if, and only if, for every $b\in\mathscr{B}(x,\Gamma_S)$ and every $i\in\{i_x^{\mathrm{sp}}+1,\dots,r\}$, we have $\partial_b\mathcal{R}_{S,i}(-,\mathscr{F})=0$.

Proof. By definition $\chi^o(x,\Gamma_S,\mathscr{F}) = r\cdot\chi(x,S) - \sum_{b\in\mathscr{B}(x,\Gamma_S)}\mathrm{Irr}_b(\mathscr{F})$. We have

$$-\sum_{b\in\mathscr{B}(x,\Gamma_S)}\mathrm{Irr}_b(\mathscr{F}) \;=\; \sum_{b\in\mathscr{B}(x,\Gamma_S)}\deg(b)\cdot\partial_b H_{S,r}(x,\mathscr{F}) \tag{2.41}$$

$$= \sum_{b\in\mathscr{B}(x,\Gamma_S)}\deg(b)\cdot\partial_b H_{S,i_x^{\mathrm{sp}}}(x,\mathscr{F}) + \sum_{b\in\mathscr{B}(x,\Gamma_S)}\sum_{j=i_x^{\mathrm{sp}}+1}^{r}\deg(b)\cdot\partial_b\mathcal{R}_{S,j}(x,\mathscr{F}) \tag{2.42}$$

The second term is C, while the first can be written as

$$\sum_{b\in\mathscr{B}(x,\Gamma_S)}\deg(b)\cdot\partial_b H_{S,i_x^{\mathrm{sp}}}(x,\mathscr{F}) \;=\; dd^cH_{S,i_x^{\mathrm{sp}}}(x,\mathscr{F}) - \sum_{b\in\mathscr{B}(x,\Gamma_S(\mathscr{F})-\Gamma_S)}\deg(b)\cdot\partial_b H_{S,i_x^{\mathrm{sp}}}(x,\mathscr{F}) \tag{2.43}$$

By Theorem 2.1.1, we have $dd^cH_{S,i_x^{\mathrm{sp}}}(x,\mathscr{F}) = -i_x^{\mathrm{sp}}\cdot\chi(x,S)$, hence we obtain (2.40).

Let $b\in\mathscr{B}(x,\Gamma_S(\mathscr{F})-\Gamma_S)$. It is the germ of segment at the open boundary of a virtual open disk D_b that is a connected component of $X-\{x\}$. By [PP13, Proposition 2.8.2], if D_b is endowed with the trivial pseudo-triangulation, the radii are invariant by localization to D_b and we can apply [Pul15, Theorem 3.3.4, item iii)-(c)] after localization to it. Indeed, by the definition of i_x^{sp}, for all $i\leqslant i_x^{\mathrm{sp}}$, the radii $\mathcal{R}_{S,i}(-,\mathscr{F})$ are spectral non-solvable along b. We deduce that one has

$$\partial_b H_{S,i_x^{\mathrm{sp}}}(x,\mathscr{F}) \;\geqslant\; 0. \tag{2.44}$$

We deduce that $B\leqslant 0$ and that $B=0$ precisely when the conditions of the statement hold.

Remark that, by definition of i_x^{sp}, for every $j\in\{i_x^{\mathrm{sp}}+1,\dots,r\}$, we have $\mathcal{R}_{S,j}(x,\mathscr{F})=1$. We deduce that, for every germ of segment b out of x, we have $\partial_b\mathcal{R}_{S,j}(-,\mathscr{F})\leqslant 0$. The claim follows. □

We state another related lemma dealing with the points of the boundary. It is placed here for expository reasons but relies on Theorem 4.1.1 that we will prove later. This lemma will only be used in Section 5, so no circular reasoning is involved.

Lemma 2.2.4. *Let $x\in\partial X$. Set $r:=\mathrm{rank}(\mathscr{F}_x)$ and assume that the radii of $\mathscr{F}$ are spectral non-solvable at x. Let D be a connected component of $X-\{x\}$ that is a virtual open disk and denote by b_D the corresponding branch emanating from x. Then, we have*

$$\deg(b)\cdot\partial_{b_D}H_{S,r}(x,\mathscr{F}) \;=\; \mathrm{h}^1_{\mathrm{dR}}(D,\mathscr{F}_{|D}) \;\geqslant\; 0\,. \tag{2.45}$$

In particular, we have

$$\chi(x,\Gamma_S,\mathscr{F}) \;\leqslant\; 0\,. \tag{2.46}$$

Proof. Let C_b be a virtual open annulus representing b. By [PP13, Lemma 2.8.4, item ii)], we have

$$\partial_{b_D}H_{S,r}(x,\mathscr{F}) = \partial_{b_D}H_{\emptyset,r}(x,\mathscr{F}_{|C_b}) + \mathrm{h}^0_{\mathrm{dR}}(D,\mathscr{F}_{|D}) - r. \tag{2.47}$$

Since the radii of $\mathscr{F}$ are all spectral non-solvable at x, we have $\mathrm{h}^0_{\mathrm{dR}}(D,\mathscr{F}_{|D})=0$.

By Remark **??**, iii), $\mathscr{F}_{|D}$ is free of Liouville numbers along b_D, hence, by Theorem 4.1.1, we have

$$\deg(b)\cdot(\partial_{b_D}H_{\emptyset,r}(x,\mathscr{F}_{|C_b}) - r) = \mathrm{Irr}_D(\mathscr{F}_{|D}) - r\,\chi_c(D) \tag{2.48}$$

$$= -\chi_{\mathrm{dR}}(D,\mathscr{F}_{|D}) \tag{2.49}$$

$$= \mathrm{h}^1_{\mathrm{dR}}(D,\mathscr{F}_{|D})\geqslant 0. \tag{2.50}$$

The last statement follows from the first combined with (2.38). □

2.2.2. Adapted pseudo-triangulations. Let X be a quasi-smooth K-analytic curve, Z a locally finite set of rigid points of X and $\mathcal{F}$ a differential equation on X with meromorphic singularities on Z. We set

$$Y \ := \ X - Z \qquad \text{and} \qquad \mathscr{F} \ := \ \mathcal{F}_{|Y} \ . \tag{2.51}$$

Definition 2.2.5 (Adapted pseudo-triangulation)**.** *We say that the pseudo-triangulation S of Y is adapted to $\mathscr{F}$ if, for every edge I of Γ_S (i.e. every connected component of $\Gamma_S - S$), the total height $H_{S,r}(-,\mathscr{F})$ is* log-affine *along I, where r is the rank of $\mathscr{F}$ around I.*

Remark 2.2.6. *Notice that a pseudo-triangulation of Y is a pseudo-triangulation of X if, and only if, every $z \in Z$ has a neighborhood in X that does not meet S.*

Observe also that if X is a finite curve, then Y is a finite curve if, and only if, Z is a finite set. In this case, every finite pseudo-triangulation of Y is a pseudo-triangulation of X.

The following result is a direct consequence of the finiteness results of [Pul15, PP15].

Lemma 2.2.7. *Every pseudo-triangulation of X can be enlarged into a pseudo-triangulation of Y.*

Every pseudo-triangulation S of Y can be enlarged into a pseudo-triangulation $S' \supseteq S$ that is adapted to $\mathscr{F}$ and satisfies $\Gamma_{S'} = \Gamma_S$.

If Y is a finite curve and if S is a finite pseudo-triangulation of Y, then the following are equivalent:

i) S can be enlarged into a finite *pseudo-triangulation S' adapted to $\mathscr{F}$ such that $\Gamma_S = \Gamma_{S'}$;*

ii) $\mathscr{F}$ has well-defined irregularity on Y (cf. Definition 2.1.8). □

2.2.3. Virtual local indexes and global irregularity.

In this section, we obtain an interpretation of the irregularity as a sum of virtual local indexes. Notice that the terms appearing in the formula (2.53) below constitute the right hand term of the index formula (4.3).

Proposition 2.2.8. *Let $\mathscr{F}$ be a differential equation on a curve Y. Assume that the curve Y is finite and that all the radii of $\mathscr{F}$ are spectral non-solvable at each point of ∂Y. Let S be a pseudo-triangulation of Y such that Γ_S is quasi-finite (cf. [PP13, Definition 1.1.32]) and meets every connected component of Y.*

Then the following assertions are equivalent:

i) $\mathscr{F}$ has well-defined irregularity (cf. Definition 2.1.8);

ii) there exists a finite subset F of Γ_S such that, for every $x \in \Gamma_S - F$, we have $\chi(x,\Gamma_S,\mathscr{F}) = 0$.

In this case, we have

$$\mathrm{Irr}_Y(\mathscr{F}) \ = \ \Big(\sum_{i=1}^{n} \chi_c(Y_i)\cdot \mathrm{rank}(\mathscr{F}_{|Y_i})\Big) - \sum_{x\in\Gamma_S} \chi(x,\Gamma_S,\mathscr{F}) \ , \tag{2.52}$$

where $Y_1,\dots,Y_n$ are the connected components of Y.

Moreover, if S is adapted to $\mathscr{F}$, then the previous assertions are also equivalent to

iii) there exists a finite subset F of S such that, for every $x \in S - F$, we have $\chi(x,\Gamma_S,\mathscr{F}) = 0$.

When it is satisfied, we have

$$\mathrm{Irr}_Y(\mathscr{F}) \ = \ \Big(\sum_{i=1}^{n} \chi_c(Y_i)\cdot \mathrm{rank}(\mathscr{F}_{|Y_i})\Big) - \sum_{x\in S} \chi(x,\Gamma_S,\mathscr{F}) \ , \tag{2.53}$$

Proof. By Lemma 2.2.7, there exists a pseudo-triangulation $S' \supseteq S$ that is adapted to $\mathscr{F}$ and satisfies $\Gamma_{S'} = \Gamma_S$. Since S' is adapted to $\mathscr{F}$, for each $x \in \Gamma_S - S'$, we have $\chi(x,\Gamma_{S'},\mathscr{F}) = \chi(x,\Gamma_S,\mathscr{F}) = 0$.

It follows that assertion ii) for Γ_S is equivalent to assertion iii) for S'. Moreover, when they hold, we have

$$\sum_{x\in\Gamma_S}\chi(x,\Gamma_S,\mathscr{F})=\sum_{x\in S'}\chi(x,\Gamma_{S'},\mathscr{F}),\tag{2.54}$$

hence (2.52) and (2.53) are equivalent. As a result, we may assume that S is adapted to $\mathscr{F}$ and it is enough to prove that i) and iii) are equivalent and that, when they are satisfied, formula (2.53) holds.

We can assume that Y is connected. Let r be the rank of $\mathscr{F}$. All the quantities are stable by scalar extension, so we can assume that K is algebraically closed. Let us write $\partial^o Y=\{b_1,\dots,b_n\}$. Let $C_1,\dots,C_n$ be open pseudo-annuli such that

a) Γ_{C_i} (suitably oriented) represents b_i;
b) $\Gamma_{C_i}\subseteq\Gamma_S$;
c) $C_i\cap C_j=\emptyset$ for $i\neq j$;
d) C_i contains no points of positive genus and no bifurcation points of Γ_S.

Point d) is possible because Γ_S is quasi-finite.

iii) $\Rightarrow$ i). Assume that we have a finite subset F of S such that $\chi(x,\Gamma_S,\mathscr{F})=0$ for all $x\in S-F$. Up to shrinking the C_i's, we may assume that they contain no points of F.

Let $i\in\{1,\dots,n\}$. Recall that S is assumed to be adapted to $\mathscr{F}$. In particular, if Γ_{C_i} contains no points of S, then the total height $H_{S,r}(-,\mathscr{F})$ is log-affine along Γ_{C_i}.

Let us now assume that $S\cap\Gamma_{C_i}\neq\emptyset$. By construction, for every $x\in S\cap\Gamma_{C_i}$, we have exactly two directions b_1,b_2 out of x that belong to Γ_S and, moreover, $g(x)=0$. We compute

$$0\;=\;\chi(x,\Gamma_S,\mathscr{F})\;=\;(2-2g(x)-N_S(x))\cdot r-(\mathrm{Irr}_{b_1}(\mathscr{F})+\mathrm{Irr}_{b_2}(\mathscr{F}))\;=\;-(\mathrm{Irr}_{b_1}(\mathscr{F})+\mathrm{Irr}_{b_2}(\mathscr{F}))\,.\tag{2.55}$$

It follows that $\mathrm{Irr}_{b_1}(\mathscr{F})=-\mathrm{Irr}_{b_2}(\mathscr{F})$, and hence the total height $H_{S,r}(-,\mathscr{F})$ is log-affine along Γ_{C_i} for all i. This implies that $\mathscr{F}$ has well-defined irregularity.

i) $\Rightarrow$ iii). Assume now that $\mathscr{F}$ has well-defined irregularity. By shrinking the C_i's, we may assume that the total height $H_{S,r}(-,\mathscr{F})$ is log-affine along Γ_{C_i} for all i. It follows that, for every i and every $x\in S\cap\Gamma_{C_i}$, we have $\chi(x,\Gamma_S,\mathscr{F})=0$.

Set $X':=X-\bigcup_i C_i$. The subset X' of X is compact, and hence $S\cap X'$ is finite. This proves property ii) with $F=S\cap X'$.

We now prove formula (2.53). Since $\mathscr{F}$ has well-defined irregularity, by Lemma 2.2.7, there exists a finite subset S' of Γ_S that is a pseudo-triangulation of X adapted to $\mathscr{F}$. For each $x\in\Gamma_S-S'$ we have $\chi(x,\Gamma_S,\mathscr{F})=0$, hence, replacing S by S', we can assume S is itself finite.

We have

$$\begin{aligned}\sum_{x\in S}\chi(x,\Gamma_S,\mathscr{F})\;&=\;\sum_{x\in S}\chi^o(x,\Gamma_S,\mathscr{F})-\sum_{x\in\partial X}\Delta_r(x,\mathscr{F}) &(2.56)\\
&=\;\sum_{x\in S}\Big(r\cdot\chi(x,S)-\sum_{b\in\mathscr{B}(x,\Gamma_S)}\mathrm{Irr}_b(\mathscr{F})\Big)-\sum_{x\in\partial X}\Delta_r(x,\mathscr{F}) &(2.57)\\
&=\;r\cdot\Big(\sum_{x\in S}\chi(x,S)\Big)-\sum_{x\in S}\sum_{b\in\mathscr{B}(x,\Gamma_S)}\mathrm{Irr}_b(\mathscr{F})-\sum_{x\in\partial X}\Delta_r(x,\mathscr{F}) &(2.58)\end{aligned}$$

By Lemma [PP13, Lemma 1.1.53], we have $\chi_c(X)=\sum_{x\in S}\chi(x,S)$. Moreover, we have

$$\sum_{x\in S}\sum_{b\in\mathscr{B}(x,\Gamma_S)}\mathrm{Irr}_b(\mathscr{F})\;=\;-\sum_{b\in\partial^o X}\mathrm{Irr}_b(\mathscr{F})\,.\tag{2.59}$$

Indeed, if I is a relatively compact edge of Γ_S (i.e. a connected component of $\Gamma_S - S$ that does not meet $\partial^o X$), and if b_1 and b_2 are the two germs of at its open boundary, we have $\mathrm{Irr}_{b_1}(\mathscr{F})+\mathrm{Irr}_{b_2}(\mathscr{F}) = 0$, because (by the choice of S) the total height $H_{S,r}(-,\mathscr{F})$ is log-affine along I and b_1 and b_2 have opposite orientations. We deduce that the only local irregularities that do not cancel are those of the open boundary.

Finally, we have

$$\sum_{x\in S}\chi(x,\Gamma_S,\mathscr{F}) \;=\; r\cdot\chi_c(X) - \Big(\sum_{x\in\partial X}\Delta_r(x,\mathscr{F}) - \sum_{b\in\partial^o X}\mathrm{Irr}_b(\mathscr{F})\Big)\;. \tag{2.60}$$

The claim follows. □

Remark 2.2.9. *Proposition 2.2.8 requires Γ_S to be quasi-finite or, equivalently, that the set of end-points of Γ_S is finite. We show here its necessity with an example (see also [PP13, Lemma 1.1.53]).*

Let $r,s\in\,]0,1[$ with $r<s$. Set $C:=\{s<|T|<1\}$. Let $(z_n)_{n\geqslant 0}$ be a sequence of K-rational points of C such that the sequence $(|z_n|)_{n\geqslant 0}$ is increasing with limit 1. Set

$$S := \{x_{z_n,r}\mid n\geqslant 0\}\cup\{x_{z_n,|z_n|}\mid n\geqslant 0\}. \tag{2.61}$$

It is a pseudo-triangulation of C.

We know that the trivial differential differential equation $\mathscr{O}_C$ over C has zero irregularity, but for all $x\in S$ we have

$$\chi(x,\Gamma_S,\mathscr{O}_C) \;=\; \chi(x,S) \;=\; 2-N_S(x) \;=\; \begin{cases} +1 & \text{if } x = x_{z_n,r}\;,\\ -1 & \text{if } x = x_{z_n,|z_n|}\;.\end{cases} \tag{2.62}$$

In this case, the sum $\sum_{x\in S}\chi(x,\Gamma_S,\mathscr{O}_C)$ does not converge.

Remark 2.2.10. *As mentioned in the introduction of this section, under appropriate assumptions, $\chi(x,\Gamma_S,\mathscr{F})$ is the index of an open neighborhood of x. We here precise those assumptions. Let S be a pseudo-triangulation adapted to $\mathscr{F}$ and let U_x be an elementary neighborhood of x in X adapted to $\mathscr{F}$ such that $U_x\cap S=\{x\}$. Denote by $\mathscr{B}(x,-\Gamma_S)$ the set of germs of segment out of x that are not in Γ_S. For every $b\in\mathscr{B}(x,-\Gamma_S)$, let $D_{x,b}$ be the virtual open disk in X containing b whose relative boundary in X is x. Set*

$$U_x(S) \;=\; U_x\cup\bigcup_{b\in\mathscr{B}(x,-\Gamma_S)} D_{x,b}\;. \tag{2.63}$$

It is clear that $\Gamma_S\cap U_x(S)$ equals the skeleton $\Gamma_{U_x(S)}$ of $U_x(S)$ and that

$$\chi(x,\Gamma_S,\mathscr{F}) \;=\; \chi_c(U_x(S))\cdot\mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_{U_x(S)}(\mathscr{F})\;. \tag{2.64}$$

Assume that $\mathscr{F}$ is strongly free of Liouville numbers along the germs of segments out of x, and that if x is a boundary point of $U_x(S)$, then the radii of $\mathscr{F}$ are all spectral non solvable at x. In this case, we will see in Theorem 4.1.1 that the right hand term of (2.64) equals the index $\chi_{\mathrm{dR}}(U_x(S),\mathscr{F})$.

Notice that the Liouville condition is automatic if all the radii of $\mathscr{F}$ are spectral non-solvable at x or if the characteristic of $\widetilde{K}$ is 0.

In presence of Liouville numbers, the index $\chi_{\mathrm{dR}}(U_x(S),\mathscr{F})$ can be infinite, but the virtual local index $\chi(x,\Gamma_S,\mathscr{F})$ is always finite. It is a local numerical invariant of $\mathscr{F}$ that seems not necessarily related to any de Rham index.

2.2.4. The meromorphic case. Let X be a finite curve, $Z\subset X$ be a finite set of rigid points and $\mathcal{F}$ a differential equation on X with meromorphic singularities on Z. We set $Y:=X-Z$ and

$\mathscr{F} := \mathcal{F}_{|Y}$

In Section 2.1.1 we have defined the irregularity of $\mathcal{F}$ on X as that of $\mathscr{F}$ on Y:

$$\mathrm{Irr}_X(\mathcal{F}) \;=\; \mathrm{Irr}_Y(\mathscr{F})\;. \tag{2.65}$$

Now, the virtual local indexes $\chi(x,\Gamma_S,\mathscr{F})$, are associated with $\mathscr{F}$ and a pseudo-triangulation S of Y. In this case, for all $x\in S$, we are allowed to set

$$\chi(x,\Gamma_S,\mathcal{F}) \;:=\; \chi(x,\Gamma_S,\mathscr{F})\;. \tag{2.66}$$

We then obtain immediately the extension of the results of Section 2.2 to the meromorphic case.

Remark 2.2.11. *Notice that we do not necessarily require S to be also a pseudo-triangulation of X. Notice that this happens if, and only if, every point of Z has a neighborhood in X without points of S.*

2.2.5. The overconvergent case. Let X be a quasi-smooth K-analytic curve, Z a locally finite set in X and $\mathcal{F}$ an overconvergent differential equation over X with meromorphic singularities at Z. We retain the notations of Section 2.1.2. We set $Y := X-Z$, $Y' := X'-Z$, $\mathscr{F} := \mathcal{F}_{|Y}$ and $\mathscr{F}' := \mathcal{F}'_{|Y'}$. By definition, $\mathscr{F}$ is also overconvergent (on Y').

Observe that $\partial X = \partial Y$ and that $Y'-Y = X'-X$. In particular, if U is an elementary neighborhood of X in X', then $U-Z$ is an elementary neighborhood of Y in Y'. Conversely, if V is an elementary neighborhood of Y in Y', then $V\cup Z$ is an elementary neighborhood of X in X'.

Moreover, an elementary neighborhood U of X in X' is adapted to $\mathcal{F}'$ if, and only if, $U-Z$ is adapted to $\mathscr{F}'$ (cf. Definition 2.1.17).

Now, we consider a pseudo-triangulation S of Y *that is adapted to* $\mathscr{F}$ (cf. Definition 2.2.5) and meets every connected component of Y.[5] In particular, S is not empty. Notice that S is also a pseudo-triangulation of every elementary neighborhood of Y in Y'.

For each $x\in\partial Y$, we denote by $b_{x,1},\ldots,b_{x,t_x}$ the germs of segments out of x that belong to Y' but not to Y. Since $\partial X = \partial Y$ and $X'-X = Y'-X$, $b_{x,1},\ldots,b_{x,t_x}$ are also the germs of segments out of x that belong to X' but not to X.

Definition 2.2.12. *Let $x\in S$. Set $r := \mathrm{rank}(\mathscr{F}_x)$. We set*

$$\chi^{\dagger}(x,\Gamma_S,\mathscr{F}) := \begin{cases} \chi(x,\Gamma_S,\mathscr{F}) - r\cdot t_x - \sum_{j=1}^{t_x}\mathrm{Irr}_{b_{x,j}}(\mathscr{F}') & \text{if } x\in\partial Y\;;\\ \chi(x,\Gamma_S,\mathscr{F}) & \text{if } x\in S-\partial Y\;, \end{cases} \tag{2.67}$$

where, if $x\in\partial Y$, $\chi(x,\Gamma_S,\mathscr{F})$ is computed with respect to Y (and not Y').

Lemma 2.2.13. *Let U be an elementary neighborhood of Y in Y' that is adapted to $\mathscr{F}'$. For all $x\in S$, we have*

$$\chi(x,\Gamma_S,\mathscr{F}'_{|U}) = \chi^{\dagger}(x,\Gamma_S,\mathscr{F}). \qquad \square \tag{2.68}$$

Proposition 2.2.14. *Assume that the curve Y is a finite curve. Let S be a pseudo-triangulation of Y such that*

i) S is adapted to $\mathscr{F}$;

ii) S meets all connected components of Y;

iii) Γ_S is quasi-finite (cf. [PP13, Definition 1.1.32]).

[5] Notice that X is connected if, and only if, so is Y. More generally, there is a canonical bijection between the set of connected components of Y and X.

Then, there exists a finite subset $F \subseteq S$ such that, for every subset $S' \subseteq S$ containing F, we have

$$\sum_{x\in S'} \chi^{\dagger}(x,\Gamma_S,\mathscr{F}) \;=\; \Big(\sum_{i=1}^{n}\chi_c(Y_i^{\dagger})\,\mathrm{rank}(\mathscr{F}_{|Y_i})\Big) - \mathrm{Irr}_{Y^{\dagger}}(\mathscr{F})\;, \tag{2.69}$$

where $Y_1,\ldots,Y_n$ are the connected components of Y.

Proof. Let U be an elementary neighborhood of Y in Y' that is adapted to $\mathscr{F}'$. As usual, S is a pseudo-triangulation of U. By Lemma 2.2.13, for all $x \in S$, we have $\chi^{\dagger}(x,\Gamma_S,\mathscr{F}) = \chi(x,\Gamma_S,\mathscr{F}'_{|U})$. Hence, by Proposition 2.2.8, there exists a finite subset F of S such that, for every subset $S' \subseteq S$ containing F, we have $\sum_{x\in S'}\chi(x,\Gamma_S,\mathscr{F}'_{|U}) = \chi_c(U) - \mathrm{Irr}_U(\mathscr{F}'_{|U})$. The claim then follows from the equalities $\chi_c(U) = \chi_c^{\dagger}(Y)$ and $\mathrm{Irr}_{Y^{\dagger}}(\mathscr{F}) = \mathrm{Irr}_U(\mathscr{F}'_{|U})$. □

3. Essential algebraizability

The notion of algebraizability is not uniform in the literature and, in order to avoid ambiguities, we begin by giving the following definition. Recall that a curve X is *finite* if it has finitely many connected components and if it admits a finite pseudo-triangulation (cf. Definition **??**).

Definition 3.0.1 (Algebraizability)**.** *Let X be a finite curve together with an open embedding into a connected projective curve P. Let Z be a finite set of rigid points in X and let $\mathcal{F}$ be a differential equation with meromorphic singularities at Z. We say that $\mathcal{F}$ is* algebraizable in P *if there exist a finite set of rigid points $Z' \subset P$ with $Z' \cap X = Z$ and a differential equation $\mathcal{F}'$ on P with meromorphic singularities at Z' together with an isomorphism*

$$\mathcal{F}'_{|X} \;\cong\; \mathcal{F}\;. \tag{3.1}$$

We also simply say that $\mathcal{F}$ is algebrizable if P is clear from the context, or if $\mathcal{F}$ is algebrizable with respect to some unspecified P.

We also recall REFthat a differential equation (without singularities) on a finite curve Y is called *finite* if it has log-affine radii at the open boundary of Y. The notion of finiteness is related to that of algebraicity because, by Lemma **??**, the extended equation $\mathcal{F}'$ has affine radii around every singularity of Z' but also over each germ of segment of $P - Z'$, by relative compactness. In particular, the radii of $\mathcal{F}$ are log-affine along the open boundary of X, and the restriction of $\mathcal{F}$ to $X - Z$ is a finite differential equation.

Unfortunately, the finiteness of $\mathcal{F}$ is not sufficient to guarantee the existence of $\mathcal{F}'$. There are indeed several examples of finite differential equations that are not algebrizable. One of the reasons is that if $\mathcal{F}$ is algebrizable, then $\mathrm{End}(\mathcal{F})$ is also algebrizable (since $\mathrm{End}(\mathcal{F}')_{|X} = \mathrm{End}(\mathcal{F})$) and it has to be a finite differential equation too. However, the affinity of the radii of $\mathcal{F}$ does not imply the same property for $\mathrm{End}(\mathcal{F})$. We illustrate this pathology in the following example.

Example 3.0.2. *Let $D = \{|T| < 1\}$ be the open unit disk, let $X = \{0 < |T| < 1\}$ and let $P = \mathbb{P}^{1,\mathrm{an}}_K$. For all ρ, let x_ρ denote the point of P associated to the sup-norm on the disk $\{|T| \leqslant \rho\}$.*

Let f be a bounded analytic function on D with infinitely many zeros. Denote by $a = \|f\|_X = \|f\|_D$ its sup-norm and assume that $a > 1$. Let $\mathscr{G}$ be the rank-one differential module over X defined by the equation $\frac{d}{dT} - f(T)$. If X is endowed with the empty pseudo-triangulation, Young's theorem (cf. Proposition **??***) gives the existence of some $\varepsilon < 1$ such that for all $\rho \in]\varepsilon, 1[$ one has $\mathcal{R}_{\emptyset,1}(\mathscr{G}, x_\rho) = \omega\cdot\rho^{-1}\cdot|f(x_\rho)|^{-1}$, (cf. (0.1)). By the choice of f, the (first) radius of $\mathscr{G}$ has infinitely many breaks as ρ approaches 1. In particular $\mathscr{G}$ is not algebrizable.*

Now, let $e \in K$ and consider the differential module $\mathscr{N}(e)$ over X defined by the equation $\frac{d}{dT} - \frac{e}{T}$.

If $|e| > a$, again by Young's theorem, for every $\rho > 0$ we have $\mathcal{R}_{\emptyset,1}(\mathscr{N}(e), x_\rho) = \omega/|e| < \omega/a = \lim_{\rho\to 1^-} \mathcal{R}_{\emptyset,1}(\mathscr{G}, x_\rho)$. Hence, for every $\rho < 1$ that is close enough to 1, we have

$$\mathcal{R}_{\emptyset,1}(\mathscr{N}(e)\otimes\mathscr{G}, x_\rho) \;=\; \mathcal{R}_{\emptyset,1}(\mathscr{N}(e), x_\rho) \;=\; \omega/|e|\;. \tag{3.2}$$

The equation $\mathscr{L} := \mathscr{N}(e)\otimes\mathscr{G}$ is then finite over X, and so is the equation $\mathscr{F} := \mathscr{N}(e)\oplus\mathscr{L}$. However

$$\mathrm{End}(\mathscr{F}) \;\cong\; (\mathscr{N}(e)\oplus\mathscr{L})\otimes(\mathscr{N}(e)\oplus\mathscr{L})^* \;\cong\; \mathscr{O}_X^2\oplus\mathscr{G}\oplus\mathscr{G}^* \tag{3.3}$$

has non-affine radii at the open boundary of D. Therefore, $\mathscr{F}$ is not algebrizable.

Similar constructions show that $\mathrm{End}(\mathscr{F})$ may have arbitrary exponents even if those of the Robba part of $\mathscr{F}$ are non-Liouville. For instance, let λ be a Liouville number satisfying $|\lambda| = 1$ and let $e' := e - \lambda$. Since $|e| > 1$, we have $|e| = |e'|$. With the above notation define $\mathscr{F}' := (\mathscr{N}(e')\otimes\mathscr{G})\oplus(\mathscr{N}(e)\otimes\mathscr{G})$. As above, the radius of $(\mathscr{N}(e')\otimes\mathscr{G})$ at the open boundary is given by (3.2) while the radii of $\mathscr{F}'$ are the union of those of $(\mathscr{N}(e')\otimes\mathscr{G})$ and $(\mathscr{N}(e)\otimes\mathscr{G})$. Therefore, for every $\rho < 1$ close enough to 1 one has

$$\mathcal{R}_{\emptyset,1}(\mathscr{F}', x_\rho) \;=\; \mathcal{R}_{\emptyset,2}(\mathscr{F}', x_\rho) \;=\; \omega/|e|\;. \tag{3.4}$$

*In particular, the radii of $\mathscr{F}'$ are both constant and spectral non-solvable at the open boundary of X. Hence, the restriction of $\mathscr{F}'$ at the open boundary has no Robba part. Therefore, by definition, $\mathscr{F}'$ has no exponent and it is hence free of Liouville numbers (cf. section **??**). However*

$$\mathrm{End}(\mathscr{F}') \;=\; \mathscr{O}_X^2\oplus\mathscr{N}(\lambda)\oplus\mathscr{N}(-\lambda) \tag{3.5}$$

is of Robba type with Liouville exponents λ and $-\lambda$.

To bypass this issue, one possible solution consists in imposing the finiteness (i.e. the affinity of the radii at the open boundary) of both $\mathcal{F}$ and $\mathrm{End}(\mathcal{F})$ over X, and more generally of any other construction in the Tannakian category generated by $\mathcal{F}$. Although this is certainly necessary for the algebraicity of $\mathcal{F}$, it is unclear whether this is equivalent to it.

Here, we introduce a weaker condition called *essential algebraizability* that holds for a large class of differential equations including a large family of differential equations with *non-affine radii at the boundary*. Roughly speaking, $\mathcal{F}$ is essentially algebrizable if it becomes algebrizable *up to restricting the open boundary of X* (i.e. up to removing some small pseudo-annuli at the open boundary of X).

From a cohomological point of view this will be enough to obtain our results in Section 4. Indeed, the general strategy consists in computing the cohomology of $\mathcal{F}$ from that of its restriction to a suitable family of sub-curves $(X_n)_n$ of X via Theorem 1.4.9.

In this section we prove that, under appropriate Liouville conditions on the exponents of $\mathrm{End}(\mathcal{F})$ at the open boundary of X, $\mathcal{F}$ is essentially algebrizable.

Definition 3.0.3 (Essential Algebraizability)**.** *Let X be a finite curve with no boundary, Z be a finite set of rigid points of X and $\mathcal{F}$ be a differential equation with meromorphic singularities at Z.*

We say that $\mathcal{F}$ is essentially algebrizable *if for any relatively compact open subset X' of X the restriction of $\mathcal{F}$ to X' is algebrizable in some projective curve P' containing X' (cf. Definition 3.0.1).*

The following lemma provides an equivalent definition.

Lemma 3.0.4. *Let $X, Z, \mathcal{F}$ be as in Definition 3.0.3. Let $(X_n)_{n\in\mathbb{N}}$ be an increasing sequence of open relatively compact subsets of X such that $\bigcup_n X_n = X$. Then, $\mathcal{F}$ is essentially algebrizable if, and only if, there exists n_0 such that, for all $m \geqslant n_0$, the restriction $\mathcal{F}_{|X_m}$ is algebrizable.*

Proof. We maintain the notations of Definition 3.0.3. It is clear that if $X' \subset X''$ are two relatively compact open of X, and if $\mathcal{F}_{|X''}$ is algebrizable in P, then so is $\mathcal{F}_{|X'}$. The claim then follows from the fact that for each X' as above there exists $m \geqslant n_0$ such that $X' \subseteq X_m$. Indeed, $\bigcup_n X_n$ is an open covering of the closure of X' in X, which is a compact topological space. □

3.1. Main result: essential algebraizability of differential equations that are free of Liouville numbers at the open boundary

Let X be a finite curve (cf. Definition **??**). We begin by introducing the main assumption of this section: a Non-Liouville condition on $\mathrm{End}(\mathcal{F})$ an the open boundary of X called (End-NL).

Definition 3.1.1 ((End-NL) condition)**.** *Let X be a finite curve without boundary. We say that $\mathcal{F}$ satisfies the condition (End-NL) on X if for all germ of segment b at the open boundary of X, the equation* $\mathrm{End}(\mathcal{F})$ *is free of Liouville numbers at b (cf. Definition* **??***).*

Lemma 3.1.2. *Let X be a finite curve and let X' be a relatively compact open subset of X. Assume that X' is a finite curve. Then there exists an open embedding of X' into the analytification P of a smooth projective curve such that $P - X'$ is a disjoint union of virtual closed disks.*

Proof. Since X' is relatively compact in X, its open boundary is represented by virtual open annuli $C_1, \ldots, C_n$ (see [PP13, Lemma 1.1.28]). Therefore, for each n, we can glue to X' a virtual open disk along C_n in order to obtain a compact curve P' without boundary containing X' as an open subset. By [Duc, Théorème 3.7.2], P' is the analytification of a smooth projective curve and the result follows.

□

Lemma 3.1.3. *Let X be a connected finite curve without boundary embedded into the analytification P of a smooth projective curve P. If $g(X) = g(P)$, then each connected component of $P - X$ is a point of type 1 or 4 or a virtual closed disk.*

Proof. Let E be a connected component of $P - X$. Since P is connected, its topological boundary contains at least one point and it cannot contains more since $g(X) = g(P)$. Let us denote this boundary point by x_E. If it has type 1 or 4, then $E = \{x_E\}$. If it has type 2 or 3, then it follows from the structure of smooth K-analytic curves that E is a virtual closed disk. □

The main result of this section is then the following

Theorem 3.1.4 (Essential algebraizability)**.** *Let X be a connected finite curve without boundary together with an open embedding into the analytification P of a smooth projective curve with $g(P) = g(X)$. Let $Z \subset X$ be a finite set of rigid points and let $\mathcal{F}$ be a differential equation over X with meromorphic singularities at Z that satisfies the condition (End-NL).*

Then, for each relatively compact open subset $X' \subset X$, there exists a finite extension L/K such that $(\mathcal{F}_L)_{|X'_L}$ is algebrizable in P_L: there exists a finite set of rigid points $Z' \subset P_L$ with $Z' \cap X'_L = Z_L \cap X'_L$ and a differential equation $\mathcal{F}'$ on P_L with meromorphic poles at Z' such that

$$\mathcal{F}'_{|X'_L} \;\cong\; (\mathcal{F}_L)_{|X'_L}\,. \tag{3.6}$$

Remark 3.1.5. *i) Notice that the radii of $\mathcal{F}$ and* $\mathrm{End}(\mathcal{F})$ *are not required to be* log-*affine at the open boundary of X and that no assumptions are made about the cohomology of $\mathcal{F}$.*

ii) The last part of the claim implies that the set $Z' - Z_L$ is in bijection with the open boundary of X_L, or equivalently with the connected components of $P_L - X_L$. However, notice that some *of these connected components may be points of type 4 (this is one of the reasons why we have to pass from X to X').*

The proof of Theorem 3.1.4 will be the object of Sections 3.2, 3.3, 3.4 and 3.5 and begins with several reductions. In Section 3.2 we show that it is enough to work locally at the open boundary of X and we reduce to the case where X is a standard pseudo-annulus. In Sections 3.3 and 3.4 we reduce to the case of an *absolutely* irreducible and then *completely* irreducible differential module. Finally, in Section 3.5 we prove the essential algebraicity in the case of a completely irreducible differential module.

The proofs of this section are generalizations to the framework of Berkovich K-analytic curves of those of [CM01], where analogous results were obtained with several restrictions about the curve, the equations and the base field. In particular, we point out a major difference with respect to [CM01], which consists in the fact that, in [CM01], the radii of $\mathcal{F}$ and $\mathrm{End}(\mathcal{F})$ are supposed affine at the open boundary of X and satisfying a certain solvability assumption.

Before embarking on the proof, let us state some corollary and remarks that illustrates the condition (End-NL) in some situation.

Corollary 3.1.6. *Let X be a finite curve, Z a finite set of rigid points of X and $\mathcal{F}$ a differential equation over X with meromorphic singularities at Z.*

If the rank of $\mathcal{F}$ is one, then $\mathcal{F}$ is essentially algebrizable.

Proof. One has $\mathrm{End}(\mathcal{F}) = \mathscr{O}_X \cdot \mathrm{Id}$ which is the trivial differential equation, therefore (End-NL) holds for $\mathcal{F}$. □

Remark 3.1.7. *Let b be a germ of segment at the open boundary of X and let C_b be an open pseudo-annulus such that Γ_{C_b} represents b. Assume that $\mathcal{F}_{|C_b}$ is extension of rank one differential equations $\{\mathcal{F}_k\}_{k=1,\ldots,r}$. We want to make more explicit in this case the condition expressed by Definition 3.1.1.*

*The Liouville condition on the exponents is stable by scalar extension (cf. Remarks **??** and **??**), therefore without loss of generality, we can assume that $K = \widehat{K^{\mathrm{alg}}}$ and that $C_b = \{r_1 < |T| < r_2\}$ is a standard pseudo-annulus.*

In this case, each $\mathcal{F}_k$ is represented by an equation $\frac{d}{dT}(Y) = g_k(T) \cdot Y$, with $g_k = \sum_{n\in\mathbb{Z}} a_{k,n}T^n$, and we can consider the residue *$\lambda_k := a_{k,-1}$ of g_k.*

Over C_b, the differential module $\mathrm{End}(\mathcal{F}) = \mathcal{F} \otimes \mathcal{F}^*$ is successive extension of the rank one equations $\{\mathcal{F}_i \otimes \mathcal{F}_j^*\}_{i,j=1\ldots,r}$ that are represented by the equations $\frac{d}{dT}(Y) = (g_i(T) - g_j(T)) \cdot Y$.

Let C'_b be a sub-pseudo-annulus of C_b along which $\mathrm{End}(\mathcal{F})$ has log-affine radii. The Robba part $\mathrm{End}(\mathcal{F}_{|C'_b})^{\mathrm{Robba}}$ is successive extension of those $(\mathcal{F}_i \otimes \mathcal{F}_j^*)_{|C'_b}$ that are of type Robba.

A criterion to detect the fact that $(\mathcal{F}_i \otimes \mathcal{F}_j^*)_{C'_b}$ is of type Robba can be deduced from [Pul07, Section 4.3]. [6]

Finally, $\mathrm{End}(\mathcal{F}_{|C'_b})$ is Free of Liouville numbers along $\Gamma_{C'_b}$ if and only if the following conditions are fulfilled:

i) for all pair (i, j) such that $(\mathcal{F}_i \otimes \mathcal{F}_j^*)_{|C'_b}$ is of Robba type over C'_b, the difference $\lambda_i - \lambda_j \in K$ is non Liouville;

[6] In [Pul07, Section 4.3] there is a criterion describing the solvability over a Robba ring. In order to find a criterion to detect the Robba property it is enough to combine the result of [Pul07, Section 4.3] with the fact that the radius of convergence function of $\mathcal{F}_i \otimes \mathcal{F}_j^*$ is a concave function on Γ_{C_b}, so that the solvability on two points of Γ_{C_b} implies the solvability on the whole segment connecting them.

ii) *for all pair* $((i,j),(i',j'))$ *such that* $(\mathcal{F}_i \otimes \mathcal{F}_j^*) \otimes (\mathcal{F}_{i'} \otimes \mathcal{F}_{j'}^*)^*_{|C'_b}$ *is of Robba type over* C'_b*, the difference* $(\lambda_i - \lambda_j) - (\lambda_{i'} - \lambda_{j'}) \in K$ *is non Liouville.*

Corollary 3.1.8. *With the notations of the above remark, if, for all* $b \in \partial^o X$*,* $\mathcal{F}_{|C_b}$ *is quasi-unipotent (i.e. successive extension of the trivial equation), or if more generally it is successive extension of rank one equations whose residues* $\{\lambda_k\}_k$ *are algebraic over* $\mathbb{Q}$*, then* $\mathcal{F}$ *satisfies (End-NL).*

We describe another interesting situation in the following results.

Corollary 3.1.9. *Let* D *be a virtual open disk endowed with the empty pseudo-triangulation. Let* $\mathscr{F}$ *be a differential equation on* D*. If the radii of* $\mathrm{End}(\mathscr{F})$ *are all constant over* D*, then* $\mathscr{F}$ *is essentially algebrizable.*

Proof. The differential equation $\mathrm{End}(\mathscr{F})$ admits a decomposition by the radii over the whole D, and this decomposition commutes with localization to any domain of D. Therefore, it makes sense to consider the Robba part of $\mathrm{End}(\mathscr{F})$ over the whole D. This Robba part is necessarily trivial, because any differential equation over D with maximal radii is so. In particular, (End-NL) holds for $\mathscr{F}$. □

3.2. Reduction to the case of a standard open annulus

We begin by a more elementary claim over a standard annulus and we prove in Lemma 3.2.2 that Theorem 3.1.4 follows from it.

Proposition 3.2.1. *Let* $C = \{r_1 < |T| < r_2\}$*,* $0 < r_1 < r_2 < +\infty$*, be a standard open annulus. Let* $\mathscr{F}$ *be a differential equation over* C *(with no meromorphic singularities) such that*

(a) the radii of both $\mathscr{F}$ *and* $\mathrm{End}(\mathscr{F})$ *are all log-affine along* Γ_C*;*

(b) $\mathrm{End}(\mathscr{F})$ *is free of liouville numbers along* Γ_C*.*

Then, for each relatively compact open sub-annulus C' *of* C *with* $\Gamma_{C'} \subset \Gamma_C$*, there exists a finite extension* L/K *and a differential equation* $\mathcal{F}'$ *over* $\mathbb{P}^{1,\mathrm{an}}_L$ *with meromorphic singularities at* $Z = \{0, \infty\}$ *such that*

$$\mathcal{F}'_{|C'_L} \cong \mathscr{F}_{|C'_L} . \tag{3.7}$$

As before, notice that the radii of $\mathscr{F}$ are not required to be log-affine and that no assumptions are made about the cohomology of $\mathscr{F}$.

Lemma 3.2.2. *Proposition 3.2.1 implies Theorem 3.1.4.*

Proof. Let $X, P, Z, \mathcal{F}$ be as in Theorem 3.1.4.

Let X' be an open relatively compact subset of X. For each $b \in \partial^o X$, we fix a pseudo-annulus C_b whose skeleton represents b such that $C_b \cap (Z \cup X') = \emptyset$. We may assume that the different C_b's do not intersect.

By assumption, for each $b \in \partial^o X$, there exists a relatively compact open sub-pseudo-annulus $C_{b,0}$ of C_b with $\Gamma_{C_{b,0}} \subset \Gamma_{C_b}$ such that $\mathrm{End}(\mathcal{F})_{|C_{b,0}}$ has log-affine radii along $\Gamma_{C_{b,0}}$ and it is free of Liouville numbers along it.

Let $I_{b,0} := \Gamma_{C_{b,0}}$ be the skeleton of $C_{b,0}$. Let $I'_{b,0}$ be a relatively compact open sub-interval of $I_{b,0}$. We denote by $C'_{b,0}$ the open sub-pseudo-annulus of $C_{b,0}$ whose skeleton is $I'_{b,0}$ and by C_b^∞ the connected component of $C_b - C'_{b,0}$ containing b.

Set $X_0 := X - \bigsqcup_{b \in \partial^o X} C_b^\infty$. It is an open connected subset of X containing X' and each element of $\partial^o X_0$ has exactly one representative among the $I'_{b,0}$'s (suitably oriented).

Since $I_{b,0}$ is relatively compact in X, $C_{b,0}$ is a virtual open annulus and it becomes a finite disjoint union of standard open annuli over a finite extension of K. There are a finite number of germs of segments at the open boundary of X so, up to replacing K by a finite extension and using Remarks **??** and **??**, we may assume that, for each b, $C_{b,0}$ is an open annulus. In this case, for each b, $C'_{b,0}$ is an open annulus that lies at the open boundary of X_0. Moreover, for each b, there is only one connected component E_b of $P - X_0$ intersecting $C_{b,0}$. We set $D_b := C'_{b,0} \cup E_b$. It is a virtual open disk containing an annulus, hence an open disk.

Now, for each b, $\mathcal{F}_{|C_{b,0}}$ satisfies the assumptions of Proposition 3.2.1. Therefore, applying the proposition to $C' := C'_{b,0}$ and $C := C_{b,0}$ we obtain a finite field extension L/K and a differential equation $\mathcal{F}'_b$ over $(D_b)_L$ having at most a meromorphic singularity at a single L-rational point of $(D_b)_L$ such that

$$(\mathcal{F}'_b)_{|(C'_{b,0})_L} \;\cong\; \mathcal{F}_{|(C'_{b,0})_L}\ . \tag{3.8}$$

Since there are finitely many b's, we can find an L such that (3.8) holds for all of them.

The curve $(X_0)_L$ together with the disks $(D_b)_L$ form an open covering of P_L with no triple intersections. Hence, we can glue $\mathcal{F}_{|(X_0)_L}$ with the differential equations $\mathcal{F}'_b$ along the $(C'_{b,0})_L$'s to obtain the required differential equation $\mathcal{F}'$ over P_L.

□

3.3. Reduction to the case of a free and absolutely irreducible differential module.

A differential equation $\mathscr{F}$ is said to be *irreducible* (or *simple*) if it has no non-trivial sub-objects (*i.e.* any injective morphism of differential equations $\mathscr{F}' \to \mathscr{F}$ is either an isomorphism or zero). It is said to be *absolutely irreducible* if, for every finite extension L/K, $\mathscr{F}_L$ is irreducible.

We maintain the notations of Proposition 3.2.1. In this section, we prove that we can shrink $C = \{r_1 < |T| < r_2\}$ and replace K by a finite extension so that, after these operations, $\mathscr{F}$ and *all its sub-quotients* are absolutely irreducible and free as $\mathscr{O}_C$-modules (cf. Lemma 3.3.3). Moreover, we show that this property allows to assume in Proposition 3.2.1 that $\mathscr{F}$ is absolutely irreducible (cf. Lemma 3.3.4).

We fix a relatively compact open sub-annulus C' in C such that $\Gamma_{C'} \subset \Gamma_C$.

Definition 3.3.1. *Denote by $\mathscr{C}(C', C)$ the set of open annuli C'' such that*

i) $C' \subset C'' \subseteq C$;

ii) C' is relatively compact in C''.

Lemma 3.3.2. *Conditions (a) and (b) of Proposition 3.2.1 are stable by replacing C with any other annulus in $\mathscr{C}(C', C)$ and by scalar extension of K. Moreover, they are automatically satisfied by all sub-quotients of $\mathscr{F}$.*

Proof. Firstly, (a) and (b) are stable by shrinking C by [PP15, Lemma 3.16] and Lemma **??**. They are also stable by scalar extensions of K, because so are the radii (cf. [PP15, Lemma 2.39]) the exponents (cf. Remarks **??** and **??**) and the cohomology (cf. Section **??**).

Now, we prove that these properties pass to the sub-quotients of $\mathscr{F}$.

Notice that if $\mathscr{F}'$ is a sub-quotient of $\mathscr{F}$, then $\mathrm{End}(\mathscr{F}')$ is also a sub-quotient of $\mathrm{End}(\mathscr{F})$. Indeed, $\mathrm{End}(\mathscr{F}')$ is isomorphic to $(\mathscr{F}')^* \otimes \mathscr{F}'$, which is clearly a sub-quotient of $\mathscr{F}^* \otimes \mathscr{F} \cong \mathrm{End}(\mathscr{F})$.

(a) By definition, the radii are all spectral on Γ_C. It is well known that if $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is an exact sequence, then the multi-set of spectral radii of $\mathscr{F}_2$ is the union of those of $\mathscr{F}_1$

and $\mathscr{F}_3$. If the radii of $\mathscr{F}_2$ are log-affine along Γ_S, then the radii of $\mathscr{F}_1$ and $\mathscr{F}_3$ have to be log-affine too.

(b) It is known that freeness of Liouville numbers both pass to sub-quotients (cf. Lemma **??** and item v) of Section **??**). In particular the freeness of Liouville numbers of $\mathrm{End}(\mathscr{F})$ implies that of $\mathrm{End}(\mathscr{F}')$. □

Lemma 3.3.3. *There exists a sub-pseudo-annulus $C_1 \in \mathscr{C}(C', C)$ and a finite extension L/K such that the irreducible sub-quotients of $(\mathscr{F}_{|C_1})_L$ satisfy the following properties:*

i) they are all absolutely irreducible;

ii) they are all free as $\mathscr{O}_{C_1}$-modules;

iii) they all satisfy the conditions (a) and (b) of Proposition 3.2.1.

Moreover these properties still hold replacing C_1 by any other $C_2 \in \mathscr{C}(C', C)$ contained in C_1 and L by any finite extension L'/L.

Proof. Recall that a Jordan-Hölder sequence for $\mathscr{F}$ is a sequence of sub-objects $0 = \mathscr{N}_0 \subset \mathscr{N}_1 \subset \mathscr{N}_2 \subset \cdots \subset \mathscr{N}_\ell = \mathscr{F}$ such that for all $i = 1, \ldots, \ell$ the quotient $\mathscr{N}_i/\mathscr{N}_{i-1}$ is irreducible. The category of differential equations over C is Artinian and Jordan-Hölder theorem holds. Therefore, the length $\ell(\mathscr{F}) := \ell$ of a Jordan-Hölder sequence and the multi-set of sub-quotients $\{\mathscr{N}_i/\mathscr{N}_{i-1}\}_{i=0,\ldots,\ell}$ is independent of the chosen Jordan-Hölder sequence.

Let $\overline{\mathscr{C}}$ be the set of pairs (C'', L) where L/K is a finite extension in a fixed algebraic closure of K, and $C'' \in \mathscr{C}(C', C)$. The set $\overline{\mathscr{C}}$ is ordered as follows $(C_1, L_1) \leqslant (C_2, L_2)$ if $C_1 \supseteq C_2$ and $L_1 \subseteq L_2$.

It is clear that the function $(C'', L) \mapsto \ell(\mathscr{F}_{|C'',L})$ is a non-decreasing function on $\overline{\mathscr{C}}$. Indeed, if $(C_1, L_2) \leqslant (C_2, L_2)$ then a Jordan-Hölder sequence over (C_2, L_1) induces a sequence over (C_2, L_2) that is part of a Jordan-Hölder sequence. Therefore, $\ell(\mathscr{F})$ cannot decrease by these operations.

Moreover, since $\ell(\mathscr{F})$ it is upper bounded by the rank of $\mathscr{F}$, there exists a pair (C_1, L_1) after which the function is constant. We can therefore find (C_1, L_1) such that i) and ii) are fulfilled, and iii) also holds by Lemma 3.3.2. □

Recall that, if $\mathcal{C}$ is an abelian category, $\mathrm{Ext}_{\mathcal{C}}(A, B)$ denotes the Yoneda group of extensions whose elements are equivalence classes of exact sequences $0 \to B \to \bullet \to A \to 0$ in $\mathcal{C}$.

If $d : R \to R$ is a derivation on a commutative ring R, we call $d - \mathrm{Mod}(R)$ the category of projective R-modules M endowed with a connection $\nabla : M \to M$ satisfying the Leibniz rule with respect to d.

Lemma 3.3.4. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two differential modules over $\mathbb{G}_m = \mathrm{Spec}(K[T, T^{-1}])$ and let $C' = \{r_1' < |T| < r_2'\}$ be a standard open annulus. For $i = 1, 2$, set*

$$\mathscr{F}_i \ := \ (\mathfrak{F}_i^{\mathrm{an}})_{|C'} \ . \tag{3.9}$$

Assume that the differential equation $\mathrm{Hom}(\mathscr{F}_1, \mathscr{F}_2)$ has finite-dimensional de Rham cohomology over C'. Then, the canonical map

$$\mathrm{Ext}_{d-\mathrm{Mod}(K[T,T^{-1}])}(\mathfrak{F}_1, \mathfrak{F}_2) \ \longrightarrow \ \mathrm{Ext}_{d-\mathrm{Mod}(\mathscr{O}(C))}(\mathscr{F}_1, \mathscr{F}_2) \tag{3.10}$$

is surjective.

Proof. Since $K[T, T^{-1}]$ is principal, $\mathfrak{F}_1(\mathbb{G}_m)$ and $\mathfrak{F}_2(\mathbb{G}_m)$ are free modules. Using [Ked10, Lemma 5.3.3] and the fact that $\mathbb{G}_m$ is affine, we have

$$\mathrm{Ext}_{d-\mathrm{Mod}(K[T,T^{-1}])}(\mathfrak{F}_1, \mathfrak{F}_2) \ = \ \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathfrak{F}_2^* \otimes \mathfrak{F}_1) \ . \tag{3.11}$$

Similarly, by Corollary 1.5.14 and [Ked10, Lemma 5.3.3 and Remark 5.3.4], we have

$$\mathrm{Ext}_{d-\mathrm{Mod}(\mathscr{O}(C'))}(\mathscr{F}_1,\mathscr{F}_2) \;=\; \mathrm{H}^1_{\mathrm{dR}}(C',\mathscr{F}_2^*\otimes\mathscr{F}_1)\;. \tag{3.12}$$

Recall also that $\mathfrak{F}_2^*\otimes\mathfrak{F}_1 \cong \mathrm{Hom}(\mathfrak{F}_2,\mathfrak{F}_1)$ and $\mathscr{F}_2^*\otimes\mathscr{F}_1 \cong \mathrm{Hom}(\mathscr{F}_2,\mathscr{F}_1)$. Since the restriction map $\mathscr{O}(\mathbb{G}_m) = K[T,T^{-1}] \to \mathscr{O}(C')$ has dense image, the result follows from Lemma 1.4.8, if K is not trivially valued. If K is trivially valued, by item i)-(a) of Section **??**, for any complete valued extension L of K, the differential equation $(\mathrm{Hom}(\mathscr{F}_1,\mathscr{F}_2))_L$ has finite-dimensional de Rham cohomology over C'_L and Lemma 1.4.8 applies again. □

Lemma 3.3.5. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two sub-quotients of $\mathscr{F}$. Then,* $\mathrm{Hom}(\mathscr{L}_1,\mathscr{L}_2)$ *has finite-dimensional de Rham cohomology over C'.*

Proof. We have $\mathrm{Hom}(\mathscr{L}_1,\mathscr{L}_2)\cong\mathscr{L}_1^*\otimes\mathscr{L}_2$ which is a sub-quotient of $\mathscr{F}^*\otimes\mathscr{F}\cong\mathrm{End}(\mathscr{F})$. Now, assumptions (a) and (b) of Proposition 3.2.1 ensure that $\mathrm{End}(\mathscr{F})$ has finite dimensional de Rham cohomology over C' (cf. Theorem **??** and item (i) of Section **??**). The claim then follows from Lemma 1.4.10. □

The results of this sub-section gives the following reduction.

Corollary 3.3.6. *In order to prove the result of Proposition 3.2.1 for some C', we may assume that, for each $C''\in\mathscr{C}(C',C)$, $\mathscr{F}_{|C''}$ is absolutely irreducible and free over C''.*

Proof. The required isomorphism (3.7) take place over C', therefore it is not restrictive to replace C by any other annulus in $\mathscr{C}(C',C)$.

On the other hand, Lemma 3.3.3 ensures that for all $C''\in\mathscr{C}(C',C)$ conveniently small all sub-quotients of $\mathscr{F}$ are free, absolutely irreducible, and satisfy the conditions of Proposition 3.2.1.

Now, assume that for all irreducible sub-quotient $\mathscr{G}$ of $\mathscr{F}$ there exists a finite extension L/K and a differential module $\mathfrak{G}$ over $L[T,T^{-1}]$ such that $\mathscr{G}_{|C'_L}\cong\mathfrak{G}_{|C'_L}$.

There are a finite number of sub-quotients, so we can assume that the field L is the same for all sub-quotients.

Lemma 3.3.5 ensures that the assumptions of Lemma 3.3.4 are always fulfilled by any pair of sub-quotients of $\mathscr{F}$, therefore interpreting $\mathscr{F}$ as a successive extension of its sub-quotients, we see that it can be extended to a differential module over $L[T,T^{-1}]$. □

3.4. Reduction to the case of a free completely irreducible differential module.

Definition 3.4.1. *Let X be an open, closed or semi-open pseudo-annulus. A differential module $\mathscr{F}$ over X is said to be* completely irreducible *if* $\mathrm{End}(\mathscr{F})$ *has log-affine radii along Γ_X and if*

$$\mathrm{End}(\mathscr{F})^{\mathrm{Robba}} \;=\; \mathrm{Id}\cdot\mathscr{O}_X\;. \tag{3.13}$$

In this section we show that it is enough to prove Proposition 3.2.1 in the case where $\mathscr{F}$ is completely irreducible and free as $\mathscr{O}_C$-module. More precisely, assuming that $\mathscr{F}$ is free and absolutely irreducible (cf. Corollary 3.3.6), we find a positive integer d such that, if φ_d denotes the standard ramification (cf. Section **??**), the pull-back of $\varphi_d^*(\mathscr{F})$ decomposes into completely irreducible sub-modules (cf. Proposition 3.4.8) whose essential algebraicity implies the essential algebraicity of $\mathscr{F}$ itself (cf. Corollary 3.4.9).

By Corollary 1.5.14, rather than working with locally free $\mathscr{O}_C$-modules with connection, we may work with their global sections. That is, we may cconsider projective $\mathscr{O}(C)$-modules of finite rank $\mathscr{F}(C)$ together with an endomorphism $\nabla:\mathscr{F}(C)\to\mathscr{F}(C)$ satisfying the Leibniz rule with respect to the derivation d/dT. We will do so in this section.

Completely irreducible modules enjoy the following important properties.

Lemma 3.4.2. *Let Y be an open, closed or semi-open pseudo-annulus. Let $\mathscr{F}$ be a completely irreducible differential equation over Y. Then*

i) for any arbitrary *complete valued field extension L/K, $\mathscr{F}_L$ is completely irreducible;*

ii) for any open, closed or semi-open sub-pseudo-annulus $X \subseteq Y$ such that $\Gamma_X \subseteq \Gamma_Y$, $\mathscr{F}_{|X}$ is completely irreducible;

Proof. Item i) follows immediately from the fact that the radii are invariant by scalar extension of K. Indeed, $(\mathrm{End}(\mathscr{F})^{\mathrm{Robba}})_L = \mathrm{End}(\mathscr{F}_L)^{\mathrm{Robba}}$.

Item ii) follows from the fact that $\mathrm{End}(\mathscr{F}_{|X}) = \mathrm{End}(\mathscr{F})_{|X}$, which implies that $\mathrm{End}(\mathscr{F}_{|X})^{\mathrm{Robba}} = (\mathrm{End}(\mathscr{F})_{|X})^{\mathrm{Robba}} = (\mathrm{End}(\mathscr{F})^{\mathrm{Robba}})_{|X}$ because the radii of $\mathrm{End}(\mathscr{F})$ are log-affine along Γ_Y. □

Lemma 3.4.3. *Let $\mathscr{F}$ be a completely irreducible differential equation over C. Then $\mathscr{F}$ is absolutely irreducible.*

Proof. Thanks to item i) of Lemma 3.4.2 it is enough to prove that $\mathscr{F}$ is irreducible over K.

We have $\chi_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F})) = \chi_{\mathrm{dR}}(\mathscr{O}(C)) = 0$ and we can apply Proposition 1.4.14 to obtain a decomposition $\mathscr{F} = \bigoplus_i \mathscr{Q}_i$, where for all i all irreducible sub-quotients of the $\mathscr{Q}_i$ are isomorphic each other and if $i \neq j$, then $\mathscr{Q}_i$ and $\mathscr{Q}_j$ have no common irreducible sub-quotients.

However, we notice that $\mathscr{F}$ can not split as direct sum of two submodules. Indeed, in this case we have two projectors that define two independent elements in $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}))$, which therefore has dimension greater than or equal to 2. This implies that the rank of $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$ is at least 2. However, this contradicts the fact that $\mathscr{F}$ is completely irreducible.

Therefore, all the sub-quotients of $\mathscr{F}$ are isomorphic the same irreducible differential module $\mathscr{N}$. It follows that $\mathscr{F}$ fits in an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{N} \to 0$, where the sub-quotients of $\mathscr{F}'$ are all isomorphic to $\mathscr{N}$. Hence, there is a projector of $\pi : \mathscr{F} \to \mathscr{F}$ with image $\mathscr{N}$. This shows that $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}))$ contains π and Id and it has dimension greater than or equal to 2. As before, this is a contradiction, and we must have $\mathscr{F} = \mathscr{N}$. The claim follows. □

Remark 3.4.4. *Let $\mathscr{F}$ be a differential equation over an open pseudo-annulus C. If $\mathrm{End}(\mathscr{F})$ is free of Liouville numbers along Γ_C, then $\chi(C, \mathrm{End}(\mathscr{F})) = 0$ and Proposition 1.4.14 applies to $\mathscr{F}$ which is then direct sum $\mathscr{F} = \oplus_i \mathscr{Q}_i$, where for all i all irreducible sub-quotients of the $\mathscr{Q}_i$ are isomorphic each other; and if $i \neq j$, then $\mathscr{Q}_i$ and $\mathscr{Q}_j$ have no common irreducible sub-quotients.*

3.4.1. Trace decomposition of $\mathrm{End}(\mathscr{F})$.

Let $\mathscr{F}$ be a differential equation over a (open, closed or semi-open) pseudo-annulus C which is free as $\mathscr{O}_C$-module.

Recall that the trace of an element $\varphi \in \mathrm{End}(\mathscr{F}(C))$ is defined as

$$\mathrm{tr}(\varphi) = \sum_i w_i^*(v_i) \in \mathscr{O}(C) \;, \tag{3.14}$$

where $\varphi = \sum_i v_i \otimes w_i^*$ in the canonical identification $\mathrm{End}(\mathscr{F}(C)) \cong \mathscr{F}(C) \otimes \mathscr{F}(C)^*$.

Definition 3.4.5. *Denote by $\mathrm{End}_0(\mathscr{F}(C))$ the sub-$\mathscr{O}(C)$-module of $\mathrm{End}(\mathscr{F}(C))$ formed by the endomorphisms with zero trace.*

For each $\varphi \in \mathrm{End}(\mathscr{F}(C))$, set

$$t_\varphi := \frac{1}{r}\,\mathrm{tr}(\varphi) \cdot \mathrm{Id} \in \mathrm{End}(\mathscr{F}(C)) \tag{3.15}$$

and

$$\varphi_0 := \varphi - t_\varphi \in \mathrm{End}_0(\mathscr{F}(C))\;. \tag{3.16}$$

The following lemma proves that the sum $\varphi = \varphi_0 + t_\varphi$ *defines a decomposition of differential equations. We denote by* ∇_0 *the restriction of* $\nabla_{\mathrm{End}(\mathscr{F}(C))}$ *to* $\mathrm{End}_0(\mathscr{F}(C))$

$$\nabla_0 \;:\; \mathrm{End}_0(\mathscr{F}(C)) \to \mathrm{End}_0(\mathscr{F}(C))\;. \tag{3.17}$$

Lemma 3.4.6. *The modules* $\mathrm{End}_0(\mathscr{F}(C))$ *and* $\mathscr{O}(C)\cdot\mathrm{Id}$ *are differential sub-modules of* $\mathrm{End}(\mathscr{F}(C))$ *and we have a decomposition of differential modules*

$$\mathrm{End}(\mathscr{F}(C)) \;=\; \mathrm{End}_0(\mathscr{F}(C)) \oplus \mathscr{O}(C)\cdot\mathrm{Id}\;. \tag{3.18}$$

Proof. The fact that $\mathrm{End}_0(\mathscr{F}(C))$ and $\mathscr{O}(C)\cdot\mathrm{Id}$ are differential sub-modules follows from the explicit description of the connection (see [Ked10, Section 5.3] for instance).

The direct sum comes from the decomposition $\varphi = \varphi_0 + t_\varphi$ of Definition 3.4.5. □

Note that in decomposition (3.18) the second summand is a trivial differential equation, hence, if the radii of $\mathrm{End}(\mathscr{F})$ are log-affine along Γ_C, it is always included into the Robba part of $\mathrm{End}(\mathscr{F}(C))$.

3.4.2. Exponents of $\mathrm{End}(\mathscr{F})$ and their pull-backs by standard ramification. We maintain the notations of Proposition 3.2.1 and Definition 3.3.1. We fix a relatively compact open sub-annulus C' of C with $\Gamma_{C'} \subset \Gamma_C$.

The results of this section can be regarded by themselves as a sort of structural theorems about completely irreducible modules. Therefore, we now isolate the two properties that are necessary to this section, and we will precise in each statements the assumptions we need.

(Abs_C) $\mathscr{F}$ is absolutely irreducible and free over C;

(L_C) $\mathrm{End}(\mathscr{F})$ has log-affine radii on Γ_C and it is free of Liouville numbers along it.

Notice that, by Lemma 3.3.2, condition (L_C) implies ($L_{C''}$) for all $C'' \in \mathscr{C}(C', C)$ and the same property for all sub-quotients of $\mathrm{End}(\mathscr{F})$. The same is not necessarily true for (Abs_C).

The following lemma shows that the exponent 0 appears in a unique Jordan block in the decomposition of type Fuchs of $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$ (cf. Lemma **??**).

Lemma 3.4.7. *Under the assumptions (*Abs_C*) and (*L_C*) one has:*

i) for each $k \geqslant 0$*, the spaces* $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F})\otimes\mathscr{L}_k)$ *and* $\mathrm{H}^1_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F})\otimes\mathscr{L}_k)$ *are one-dimensional, generated by* $\mathrm{Id}\otimes 1$ *and by the image of* $\frac{1}{T}\mathrm{Id}\otimes 1$ *respectively (see Section* **??***);*

ii) for i=0,1, the space $\mathrm{H}^i_{\mathrm{dR}}(C, \mathrm{End}_0(\mathscr{F}))$ *is zero (i.e. the connection is bijective on* $\mathrm{End}_0(\mathscr{F})$*);*

iii) the image of $\nabla : \mathrm{End}(\mathscr{F}) \to \mathrm{End}(\mathscr{F})$ *is the* K*-subspace of endomorphisms whose trace has residue zero.*

Proof. We prove item i) by induction on $k \geqslant 0$. For $k = 0$ we have $\mathscr{L}_0 = \mathscr{O}(C)$. By definition, the elements of $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}))$ are the endomorphisms $u : \mathscr{F}(C) \to \mathscr{F}(C)$ that commute with the connection of $\mathscr{F}(C)$. Since the composition of two such endomorphisms still commutes with the connection, the sub-K-algebra $K[u] \subset \mathrm{End}(\mathscr{F}(C))$ generated by u is contained in $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}))$. Since the latter space is finite-dimensional, the same holds for $K[u]$. Hence, there exists a monic polynomial $P \in K[X]$ such that $P(u) = 0$.

Let K' be a finite extension of K where P splits: $P = \prod_{i=1}^n (X - \lambda_i)$, for some $\lambda_1, \ldots, \lambda_n \in K'$. Denoting by $u_{K'}$ the endomorphism of $\mathscr{F}_{K'}(C_{K'})$ induced by u, we have $\prod_{i=1}^n (u_{K'} - \lambda_i\cdot\mathrm{Id}) = 0$, hence, for some $i \in \{1, \ldots, n\}$, we have $\mathrm{Ker}(u_{K'} - \lambda_i\cdot\mathrm{Id}) \neq 0$. Since this kernel is a differential sub-module of $\mathscr{F}_{K'}(C_{K'})$, it must be equal to $\mathscr{F}_{K'}(C_{K'})$, because $\mathscr{F}$ is absolutely irreducible. It follows that $u_{K'} = \lambda_i \cdot \mathrm{Id}$ and that u is also an homothety.

This shows that $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}(C))) = \mathrm{Id} \cdot K \subset \mathrm{Id} \cdot \mathscr{O}(C)$. Now, by the decomposition (3.18) it follows that $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}_0(\mathscr{F}(C))) = 0$, and $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{Id} \cdot \mathscr{O}(C)) = 1$. Since $\mathrm{End}(\mathscr{F})$ and all its sub-quotients have log-affine radii and are free of Liouville numbers, we can apply the index Theorem **??** (cf. item (i) of Section **??**). We deduce that $\mathrm{H}^1_{\mathrm{dR}}(C, \mathrm{End}_0(\mathscr{F}(C))) = 0$ too, and that $\mathrm{H}^1_{\mathrm{dR}}(C, \mathrm{Id}\cdot\mathscr{O}(C))$ is one dimensional. Notice that the map $f \in \mathscr{O}(C) \mapsto f \cdot \mathrm{Id} \in \mathrm{Id} \cdot \mathscr{O}(C)$ is an isomorphism of differential equations, therefore $\mathrm{H}^1_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}(C))) = \mathrm{H}^1_{\mathrm{dR}}(C, \mathrm{Id} \cdot \mathscr{O}(C))$ is generated by the image of $\frac{1}{T} \cdot \mathrm{Id}$.

Items ii) and iii) follows easily using the decomposition of Lemma 3.4.6.

Let us now consider the case where $k > 0$ in item i). We proceed by induction on k, the case $k = 0$ being true.

Every element of $\mathrm{End}(\mathscr{F}(C)) \otimes \mathscr{L}_k(C)$ can be uniquely written as $v = \sum_{i=0}^{k} \alpha_i \otimes \log(T)^i$, where $\alpha_i \in \mathrm{End}(\mathscr{F}(C))$. Then,

$$\nabla(v) \;=\; \nabla(\alpha_k) \otimes \log(T)^k + \sum_{i=0}^{k-1} \Big(\nabla(\alpha_i) + (i+1)T^{-1}\alpha_{i+1}\Big) \otimes \log(T)^i \; . \tag{3.19}$$

If $\nabla(v) = 0$, then $\nabla(\alpha_k) = 0$ and, for all $i = 0, \ldots, k-1$, we have

$$\nabla(\alpha_i) \;=\; -(i+1)T^{-1}\alpha_{i+1} \; . \tag{3.20}$$

Since $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}(C))) = \mathrm{Id} \cdot \mathscr{O}(C)$, we must have $\alpha_k \in K \cdot \mathrm{Id}$.

We now consider equation (3.20) in the case $i = k-1$. Since $\mathrm{H}^1_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}(C)))$ is one-dimensional generated by the image of $\frac{1}{T}\mathrm{Id}$, we deduce that the equation can be solved if and only if $\alpha_k = 0$. Indeed, otherwise $-kT^{-1}\alpha_k$ does not lie in the image of the connection of $\mathrm{End}(\mathscr{F})$. We deduce that $\nabla(\alpha_{k-1}) = 0$ and hence v belongs to $\mathrm{End}(\mathscr{F}) \otimes \mathscr{L}_{k-1} \subset \mathrm{End}(\mathscr{F}) \otimes \mathscr{L}_k$ and we conclude by induction. It follows that $\alpha_k = \alpha_{k-1} = \cdots = \alpha_1 = 0$ and $\alpha_0 \in K \cdot \mathrm{Id}$. Therefore,

$$\mathrm{H}^0_{\mathrm{dR}}(C', \mathrm{End}(\mathscr{F}) \otimes \mathscr{L}_k) \;=\; K \cdot \mathrm{Id} \; . \tag{3.21}$$

Now, by Remark **??**, $\mathrm{End}(\mathscr{F}) \otimes \mathscr{L}_k$ is free of Liouville numbers along Γ_C, hence, by the index Theorem **??**, the group $\mathrm{H}^1_{\mathrm{dR}}(C', \mathrm{End}(\mathscr{F}) \otimes \mathscr{L}_k)$ is one-dimensional too. A computation similar to the one above shows that $\nabla(v) = T^{-1} \cdot \mathrm{Id} \otimes 1$ has no solutions. The claim follows. □

The following proposition is the central result of this section.

Proposition 3.4.8. *We maintain the assumptions* (Abs_C) *and* (L_C)*. Then*

i) There exists an integer $d > 0$ and an automorphism $\alpha_{1/d} : \mathscr{F} \xrightarrow{\sim} \mathscr{F}$ of $\mathscr{O}_C$-modules (that does not necessarily commute with the connection) with the following properties:

(a) Every exponent of $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$ has multiplicity one and the set of the exponents of $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$ is

$$\mathfrak{e}(\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}) \;=\; \Big\{0, \frac{1}{d}, \ldots, \frac{d-1}{d}\Big\} \;\subset\; K/\mathbb{Z} \; . \tag{3.22}$$

In particular, the exponents form a cyclic group of order d.

(b) We have

$$\mathrm{End}(\mathscr{F})^{\mathrm{Robba}} \;=\; \bigoplus_{i=0}^{d-1} \mathscr{N}(i/d) \; , \tag{3.23}$$

where $\mathscr{N}(e)$ denotes the rank-one differential module associated with the equation $\frac{d}{dT}(y) = \frac{e}{T}y$. Moreover, for all $i \in \mathbb{Z}$, $\alpha^i_{1/d}$ is a generator of $\mathscr{N}(i/d)$ (here $\alpha^i_{1/d} = \alpha_{1/d} \circ \cdots \circ \alpha_{1/d}$

i-times).

(c) *The integer d divides r and, if the characteristic p of the residual field of K is positive, then d is prime to p : $(d,p)=1$.*

Assume that the set μ_d of d-th roots of unity in K contains exactly d elements.

Let $C^{1/d} := \{r_1^{1/d} < |\tilde{T}| < r_2^{1/d}\}$ and let $\varphi : C^{1/d} \to C$ be the morphism $x \mapsto x^d$. This is a finite étale Galois covering with Galois group isomorphic to μ_d. Then

ii) *If K contains the d-th roots of unity, there exists a possibly non free irreducible differential equation $\mathscr{M}$ over $C^{1/d}$ with the following properties:*

(a) *The rank of $\mathscr{M}$ is r/d;*

(b) *One has*

$$\varphi^*\mathscr{F} \;=\; \bigoplus_{\xi\in\mu_d} \xi^*\mathscr{M}\;, \tag{3.24}$$

and, for all $\xi_1 \neq \xi_2 \in \mu_d$, the differential modules $\xi_1^*\mathscr{M}$ and $\xi_2^*\mathscr{M}$ are not isomorphic.

(c) *For each $\xi \in \mu_d$, one has*

$$\varphi_*\xi^*\mathscr{M} \;\cong\; \mathscr{F}\;. \tag{3.25}$$

(d) *For each $\xi \in \mu_d$, $\xi^*\mathscr{M}$ is completely irreducible over $C^{1/d}$.*

(e) $\mathrm{End}(\xi^*\mathscr{M})$ *has log-affine radii and it is free of Liouville numbers along $\Gamma_{C^{1/d}}$.*

iii) *(Compatibility with restrictions). Assume that for all $C'' \in \mathscr{C}(C',C)$ conditions ($Abs_{C''}$) and ($L_{C''}$) hold. Then, all the statements still hold replacing C by any $C'' \in \mathscr{C}(C',C)$ and $\mathscr{F}$ by $\mathscr{F}_{|C''}$. Moreover, for all $C'' \in \mathscr{C}(C',C)$, the module decomposition (3.24) furnished by item ii) over C'' coincides with the restriction of the same decomposition over C. In particular, $\mathscr{M}$ remains completely irreducible after restriction to $(C'')^{1/d}$.*

Proof. We first focus on i)-(a) and i)-(c). If $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{N}$ are differential modules, then we have natural morphisms of differential modules

$$\psi_{\mathscr{N}} \;:\; \mathrm{Hom}(\mathscr{F}_1,\mathscr{F}_2)\otimes\mathscr{N} \;\xrightarrow{\sim}\; \mathrm{Hom}(\mathscr{F}_1,\mathscr{F}_2\otimes\mathscr{N}) \tag{3.26}$$

$$\psi'_{\mathscr{N}} \;:\; \mathrm{Hom}(\mathscr{F}_1,\mathscr{F}_2) \;\longrightarrow\; \mathrm{Hom}(\mathscr{F}_1\otimes\mathscr{N},\mathscr{F}_2\otimes\mathscr{N}) \tag{3.27}$$

where for $\alpha\in\mathrm{Hom}(\mathscr{F}_1,\mathscr{F}_2)$, $f\in\mathscr{F}_1$, $n\in\mathscr{N}$, we have $\psi_{\mathscr{N}}(\alpha\otimes n)(f)=\alpha(f)\otimes n$, and $\psi'_{\mathscr{N}}(\alpha)=\alpha\otimes 1$. Moreover, $\psi_{\mathscr{N}}$ is always an isomorphism, while $\psi'_{\mathscr{N}}$ is so only if $\mathscr{N}$ has rank one.

By Lemma **??** and by (3.26), we deduce that the element $e\in K$ (cf. Sections **??** and **??**) belongs to an exponent of $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$ if, and only if, there exists a non-zero morphism $\mathscr{F}\to\mathscr{F}\otimes\mathscr{N}(-e)$ of differential modules. Such a morphism is automatically an isomorphism because $\mathscr{F}$ is irreducible. Hence, in this case, the bijectivity implies that $-e$ is also an exponent.

If e' is another exponent, and if $\mathscr{F}\xrightarrow{\sim}\mathscr{F}\otimes\mathscr{N}(-e')$ is a corresponding isomorphism, then, by composition and using the identification $\psi'_{\mathscr{N}(e')}$, we easily deduce an isomorphism $\mathscr{F}\xrightarrow{\sim}\mathscr{F}\otimes\mathscr{N}(-e)\otimes\mathscr{N}(-e')=\mathscr{F}\otimes\mathscr{N}(-e-e')$. Hence, $e+e'$ is an exponent. This proves that (without counting the multiplicities) the exponents form a finite subgroup of $K/\mathbb{Z}$. Necessarily, this group is of the form $\{0,\frac{1}{d},\ldots,\frac{d-1}{d}\}$ for some integer d. If the residual field of K has characteristic $p>0$, then d is prime to p because, by definition, in that case one has $\frac{1}{d}\in\mathbb{Z}_p$ (cf. Lemma **??**).

To prove that d divides $r:=\mathrm{rank}(\mathscr{F})$, we observe that an isomorphism $\mathscr{F}\xrightarrow{\sim}\mathscr{F}\otimes\mathscr{N}(-e)$ induces an isomorphism between the top exterior powers $\wedge^r\mathscr{F}\xrightarrow{\sim}\wedge^r(\mathscr{F}\otimes\mathscr{N}(-e))$. If G is the matrix of the action of $\frac{d}{dT}$ on $\mathscr{F}$ in a basis, then $G-(\frac{e}{T})\mathrm{Id}$ is the matrix of the action of the connection on $\mathscr{F}\otimes\mathscr{N}(-e)$ in the corresponding basis. So, $\mathrm{tr}(G)$ and $\mathrm{tr}(G)-\frac{er}{T}$ are the matrices of the connections of $\wedge^r\mathscr{F}$ and $\wedge^r(\mathscr{F}\otimes\mathscr{N}(-e))$ respectively. Since they are isomorphic, $-er/T$ has to be the matrix of the trivial differential equation. Hence, $-er$ is an integer. For $e = 1/d$, this shows that d divides $\mathrm{rank}(\mathscr{F})$.

It remains to show that each exponent has multiplicity 1. By Lemma **??**, we have to compute the dimension of $\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F})^{\mathrm{Robba}} \otimes \mathscr{N}(-e) \otimes \mathscr{L}_k)$. In fact, we now prove that, for all $k \geqslant 0$, we have $\dim \mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}) \otimes \mathscr{N}(-e) \otimes \mathscr{L}_k) = 1$. By (3.26), this is equivalent to $\dim \mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{Hom}(\mathscr{F}, \mathscr{F} \otimes \mathscr{N}(-e)) \otimes \mathscr{L}_k) = 1$. Now, a non-trivial isomorphism $\alpha : \mathscr{F} \xrightarrow{\sim} \mathscr{F} \otimes \mathscr{N}(-e)$ of differential modules permits to identify $\mathrm{End}(\mathscr{F})$ with $\mathrm{Hom}(\mathscr{F}, \mathscr{F} \otimes \mathscr{N}(-e))$ by sending an endomorphism β into $\alpha \circ \beta$. Hence

$$\mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{Hom}(\mathscr{F}, \mathscr{F} \otimes \mathscr{N}(-e)) \otimes \mathscr{L}_k) \;\cong\; \mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}) \otimes \mathscr{L}_k) \tag{3.28}$$

and the claim now follows from point i) of Lemma 3.4.7.

We now prove i)-(b). The decomposition (3.23) is a consequence of i)-(a) by the structure theorem of modules of Robba type (cf. Theorem **??**, and [CM97, Corollary 6.2-6]). Now, if $\alpha_{1/d} \in \mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$ belongs to the factor $\mathscr{N}(1/d)$, we have $\nabla(\alpha_{1/d}) = \frac{1/d}{T}\alpha_{1/d}$ and hence also

$$\nabla(\alpha^i_{1/d}) \;=\; i\alpha^{i-1}_{1/d}\nabla(\alpha_{1/d}) \;=\; \frac{i/d}{T}\alpha^i_{1/d}\,, \qquad i \in \mathbb{Z}\,. \tag{3.29}$$

Therefore, for all $i \in \mathrm{Z}$, $\alpha^i_{1/d}$ generates $\mathscr{N}(i/d)$ in (3.23).

We now focus on ii). The map $\varphi : C^{1/d} \to C$ is an étale Galois covering with Galois group μ_d and the action of $\xi \in \mu_d$ is given by $\tilde{T} \mapsto \xi\tilde{T}$, where $\tilde{T}$ is the coordinate of $C^{1/d}$. Firstly, we notice that one has an isomorphism of differential modules $\mathrm{End}(\varphi^*(\mathscr{F})) \cong \varphi^*(\mathrm{End}(\mathscr{F}))$. The behavior by pull-back and push-forward by φ of the radii is well known (cf. for instance [PP15, Lemma 3.23]). In particular, the radii of $\mathrm{End}(\varphi^*\mathscr{F})$ are log-affine along Γ_C and

$$\mathrm{End}(\varphi^*\mathscr{F})^{\mathrm{Robba}} \;=\; \varphi^*((\mathrm{End}(\mathscr{F}))^{\mathrm{Robba}})\,. \tag{3.30}$$

Hence, we deduce from the decomposition (3.23) that $\mathrm{End}(\varphi^*\mathscr{F})^{\mathrm{Robba}}$ is a trivial differential module. Indeed, for all k one has $\varphi^*(\mathscr{N}(k/d)) = \mathscr{N}(0) = \mathscr{O}_{C^{1/d}}$. Therefore, the dimension of $\mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{End}(\varphi^*(\mathscr{F})))$ equals the rank of $\mathrm{End}(\varphi^*(\mathscr{F}))^{\mathrm{Robba}}$ and $\mathrm{End}(\mathscr{F})^{\mathrm{Robba}}$.

Now, we want to define an injective group morphism $\mathbb{Z}/d\mathbb{Z} \to \mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{End}(\varphi^*(\mathscr{F})))$. Equation (3.29) shows that $\nabla(T^{-1}\alpha^d_{1/d}) = 0$, therefore there exists $\lambda \in K$ such that $T^{-1}\alpha^d_{1/d} = \lambda \cdot \mathrm{Id} \in \mathrm{H}^0_{\mathrm{dR}}(C, \mathrm{End}(\mathscr{F}))$. Up to replace K by a finite extension, we may find an element $\alpha := \lambda^{-1/d}\alpha_{1/d}$ that satisfies $\alpha^d = T \cdot \mathrm{Id}$. Let $\widetilde{T} = T^{1/d} \in \mathscr{O}(C^{1/d})$ and set:

$$g \;:=\; \tilde{T}^{-1} \cdot \varphi^*(\alpha) \;=\; \alpha \otimes \tilde{T}^{-1} \in \mathrm{End}(\varphi^*\mathscr{F})\,, \tag{3.31}$$

where we have identified $\mathscr{F}$ with the differential module of its global sections and $\varphi^*\mathscr{F}$ with $\mathscr{F} \otimes_{\mathscr{O}(C)} \mathscr{O}(C^{1/d})$, in which the connection is given by $\nabla(\alpha \otimes f) = \nabla(\alpha) \otimes f + \alpha \otimes \frac{d}{dT}(f)$ (where $\frac{d}{dT} = \frac{\tilde{T}^{1-d}}{d} \cdot \frac{d}{d\tilde{T}}$, cf. (**??**)).

We have $g^d = \mathrm{Id}$ and $\nabla(g) = 0$, *i.e.* $g \in \mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{End}(\varphi^*\mathscr{F}))$. Moreover, for all $k = 1, \dots, d-1$, we have $g^k \neq \mathrm{Id}$, because every element of $\mathrm{End}(\varphi^*\mathscr{F}) = \mathrm{End}(\mathscr{F}) \otimes \mathscr{O}(C^{1/d})$ can be uniquely written as $\sum_{k=0}^{d-1} \beta_k \otimes \widetilde{T}^k$, with $\beta_k \in \mathrm{End}(\mathscr{F})$. Hence, the map $\rho : \mathbb{Z}/d\mathbb{Z} \to \mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{End}(\varphi^*(\mathscr{F})))$ given by $\rho(n) = g^n$ is an injective group morphism.

By construction, this action commutes with the connection of $\varphi^*\mathscr{F}$ and satisfies the following property with respect to the action of Galois group μ_d. For all $\xi \in \mu_d$, we denote by $[\xi] : \varphi^*\mathscr{F} \to \varphi^*\mathscr{F}$ the natural semi-linear action of the Galois group μ_d on the right factor of the tensor product $\varphi^*\mathscr{F} = \mathscr{F} \otimes \mathscr{O}(C^{1/d})$, which is given by

$$[\xi]\Big(\sum_{k=0}^{d-1} v_k \otimes \tilde{T}^k\Big) \;:=\; \sum_{k=0}^{d-1} v_k \otimes \xi^k\tilde{T}^k\,. \tag{3.32}$$

Then, for all $\xi \in \mu_d$, we have

$$g^i \circ [\xi] \;=\; \xi^i \cdot ([\xi] \circ g^i)\;, \qquad i = 0, \ldots, d-1\;. \tag{3.33}$$

Notice that in the statement the (Galois) action of ξ does not have brackets $[\cdot]$. In the proof we decided to change the notation in order to distinguish the multiplication $f \mapsto \xi \cdot f$ by the scalar ξ from its action (3.32) as Galois.

The group of characters of $\mathbb{Z}/d\mathbb{Z}$ is cyclic, of order d, generated by a character $\chi : \mathbb{Z}/d\mathbb{Z} \to K^\times$ sending 1 to a primitive d-th root ξ_d of 1. The projector on the χ^j-typical component of $\varphi^*(\mathscr{F})$ is then given by

$$\pi_{\chi^j} \;:=\; \frac{1}{d} \cdot \sum_{h \in \mathbb{Z}/d\mathbb{Z}} \chi^j(h^{-1}) h\;, \qquad j = 1, \ldots, d. \tag{3.34}$$

Since the action of $\mathbb{Z}/d\mathbb{Z}$ commutes with the connection of $\varphi^*\mathscr{F}$, the image of the projector is a possibly non free differential module $\mathscr{M}_j \subseteq \varphi^*\mathscr{F}$. We then obtain d distinct differential modules $\mathscr{M}_1, \ldots, \mathscr{M}_d$ that are characterized by the fact that $\mathbb{Z}/d\mathbb{Z}$ acts on the elements of $\mathscr{M}_j$ as $h(x) = \chi^j(h) \cdot x$, $h \in \mathbb{Z}/d\mathbb{Z}$, $x \in \mathscr{M}_j$. Moreover, since π_{χ^j} is a projector and $\sum_{j=1}^d \pi_{\chi^j} = \mathrm{Id}$, we have the direct sum decomposition (3.24).

Now, it follows from (3.33) that if $\xi = \xi_d^k$, then $[\xi] \circ \pi_{\chi^j} = \pi_{\chi^{j+k}} \circ [\xi]$. In particular, $[\xi]^*(\mathscr{M}_j) = \mathscr{M}_{j+k}$. Therefore, the Galois group μ_d permutes transitively $\mathscr{M}_1, \ldots, \mathscr{M}_d$. In particular, each $\mathscr{M}_j$ has rank r/d, where $r := \mathrm{rank}(\mathscr{F})$.

We now prove that each $\mathscr{M}_j$ is completely irreducible. The direct image $\varphi_*\mathscr{M}_j$ is contained in $\varphi_*\varphi^*\mathscr{F} \cong \bigoplus_{i=0}^{d-1} \mathscr{F} \otimes \mathscr{N}(i/d) \cong \mathscr{F}^d$. Hence the irreducible sub-quotients of $\varphi_*\mathscr{M}_j$ are all isomorphic to $\mathscr{F}$. Since the rank of $\varphi_*\mathscr{M}_j$ is r, we deduce that $\varphi_*\mathscr{M}_j \cong \mathscr{F}$. This implies that $\mathscr{M}_j$ is irreducible for all j, since a non trivial sub-object of $\mathscr{M}_j$ produces a non trivial sub-object of its push-forward $\varphi_*\mathscr{M}_j = \mathscr{F}$. Since $\mathscr{F}$ remains irreducible after finite scalar extension of K, so does $\mathscr{M}$ because its definition commutes with scalar extension. Therefore, $\mathscr{M}_j$ is absolutely irreducible.

The direct sum decomposition (3.24) implies

$$\mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{End}(\varphi^*\mathscr{F})) \;=\; \bigoplus_{j=1}^{d} \mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{End}(\mathscr{M}_j)) \oplus \bigoplus_{i \neq j} \mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{Hom}(\mathscr{M}_i, \mathscr{M}_j))\;. \tag{3.35}$$

Now, the identity being an horizontal section, for all j the dimension of $\mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{End}(\mathscr{M}_j))$ is at least one. Since $\mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{End}(\varphi^*\mathscr{F}))$ has dimension d, we must have $\mathrm{H}^0_{\mathrm{dR}}(C^{1/d}, \mathrm{Hom}(\mathscr{M}_i, \mathscr{M}_j)) = 0$ for all $i \neq j$. In other words, $\mathscr{M}_i$ and $\mathscr{M}_j$ are not isomorphic as differential modules.

Analogously, we have

$$\mathrm{End}(\varphi^*\mathscr{F})^{\mathrm{Robba}} \;=\; \bigoplus_{j=1}^{d} \mathrm{End}(\mathscr{M}_j)^{\mathrm{Robba}} \oplus \bigoplus_{i \neq j} \mathrm{Hom}(\mathscr{M}_i, \mathscr{M}_j)^{\mathrm{Robba}}\;. \tag{3.36}$$

As above, we know that $\mathrm{End}(\varphi^*\mathscr{F})^{\mathrm{Robba}}$ has rank d and that the rank of $\mathrm{End}(\mathscr{M}_j)^{\mathrm{Robba}}$ is at least 1. It follows that, for $i \neq j$, one has $\mathrm{Hom}(\mathscr{M}_i, \mathscr{M}_j)^{\mathrm{Robba}} = 0$ and that $\mathrm{End}(\mathscr{M}_j)^{\mathrm{Robba}}$ is free and it is reduced to $\mathscr{O}(C) \cdot \mathrm{Id}$. Therefore $\mathscr{M}_i$ is completely irreducible for all i.

To end the proof of ii) we observe that, for all i, $\mathrm{End}(\mathscr{M}_j)$ is a sub-quotient of $\mathrm{End}(\varphi^*\mathscr{F})$. Therefore, Lemma 3.3.2 shows that its radii are log-affine along $\Gamma_{C^{1/d}}$ and it is free of Liouville numbers. Finally, iii) is straightforward. The claim follows. □

Corollary 3.4.9. *It is enough to prove Proposition 3.2.1 for a differential equation that is completely irreducible and free as $\mathscr{O}_C$-module.* □

Proof. We have seen in Corollary 3.3.6 that we can assume that $\mathscr{F}$ is absolutely irreducible and free over C and over any $C'' \in \mathscr{C}(C', C)$. Therefore, for all C'', conditions ($\mathrm{Abs}_{C''}$) and ($L_{C''}$) at the beginning of this section are fulfilled by $\mathscr{F}$ and Proposition 3.4.8 holds over all C''.

Now, maintain the notations of Proposition 3.4.8. An explicit computation shows that push-forward by φ preserves essential algebrization (cf. [Pul13, Section 4.1]). Therefore, it is clear that if $\mathscr{M}$ is essentially algebrizable, then so is its push-forward $\mathscr{F}$.

Notice that $\mathscr{M}$ is not necessarily free over $C^{1/d}$, but it becomes free over any smaller annulus relatively compact in $C^{1/d}$. Therefore, up to shrinking C, we can assume that $\mathscr{M}$ is free. It is now clear that it is not restrictive to assume $\mathscr{F} = \mathscr{M}$ and $C = C^{1/d}$. The claim follows. □

3.5. Essential algebraicity of a completely irreducible module

In this section we prove the essential algebraicity of a completely irreducible module. Let C is a standard open annulus and $C' \subset C$ is a relatively compact open sub-annulus such that $\Gamma_{C'} \subset \Gamma_C$. As in the above section, we precise that the only property we need in the following claims is the fact that *$\mathscr{F}$ is an completely irreducible differential module over C*. In particular, as in the above sections, we do not require the affinity of the radii of $\mathscr{F}$ nor the finiteness of its cohomology. However, for all $C'' \in \mathscr{C}(C', C)$, $\mathscr{F}$ automatically satisfies the properties ($\mathrm{Abs}_{C''}$) and ($L_{C''}$) of Section 3.4.2.

In the proofs of this section we will work explicitly with the matrix of the connection of $\mathrm{End}(\mathscr{F})$. We recall that, for any (open or closed) annulus X on which $\mathscr{F}$ is free, one may compute explicitly the connection on $\mathrm{End}(\mathscr{F}(X))$ in terms of the connection on $\mathscr{F}(X)$. To do so, let us fix a basis of $\mathscr{F}(X)$ as an $\mathscr{O}(X)$-module. Then, there exists a matrix $G \in M_r(\mathscr{O}(X))$ such that the connection of $\mathscr{F}$ is given by $\nabla = \frac{d}{dT} + G$. Thanks to the basis, one may identify each element of $\mathrm{End}(\mathscr{F}(X))$ to a square matrix $M \in M_r(\mathscr{O}(X))$. The connection then acts by

$$\nabla_{\mathrm{End}(\mathscr{F}(X))}(M) = \frac{d}{dT}(M) + GM - MG\,. \tag{3.37}$$

In the following, we will fix a basis of $\mathscr{F}(X)$ and identify $\mathrm{End}(\mathscr{F}(X))$ to $M_r(\mathscr{O}(X))$.

If X is a *closed* annulus in C we will endow $\mathrm{End}(\mathscr{F}(X))$ with the norm $\|\cdot\|_X$ that is equal to the maximum of the sup-norms of the coefficients of the matrix. The space $\mathrm{End}(\mathscr{F}(X))$ then becomes a Banach space.

We also consider perturbation of the connection of $\mathscr{F}$ as follows. For each $H \in M_r(\mathscr{O}(X))$, denote by $\mathscr{F}^H(X)$ the differential equation on $\mathscr{F}(X)$ with connection

$$\nabla^H := \nabla + H = \frac{d}{dT} + G + H\,. \tag{3.38}$$

Now, we will prove that, under suitable hypotheses on H, the differential module $\mathscr{F}^H(C')$ is isomorphic to $\mathscr{F}(C')$. The isomorphism is obtained by a recursive process based on the following result.

Theorem 3.5.1 ([CM01, Théorème 4.1-1])**.** *Let A be a possibly non-commutative $\mathbb{Q}$-algebra. Let $n \geqslant 1$. Let $\|.\| : A \to \mathbb{R}_{\geqslant 0}$ be an ultrametric norm on A and let $\nabla : A \to A$ be a derivation. Let a_0 be a unit in A and let $a_1, \dots, a_n, c \in A$ such that*

$$\nabla(a_i) \;=\; a_{i-1} \cdot c\,, \qquad \textit{for all } i = 1, \dots, n\,. \tag{3.39}$$

Let $\rho > 0$ such that

$$\|i! a_i\| \;\leqslant\; \rho^i\,, \qquad \textit{for all } i = 1, \dots, n\,. \tag{3.40}$$

Let V be the $\mathbb{Q}$-subspace of A generated by the commutators $[x, y] = xy - yx$, for $x, y \in A$, and by the sub-algebra $\mathbb{Q}[a_0, \dots, a_n]$.

There exists $P_{n+1} \in \mathbb{Q}[a_0, \dots, a_n]$ *and* $R_{n+1} \in V$ *such that*

$$a_n \cdot c \;=\; \nabla(P_{n+1}) + R_{n+1} \tag{3.41}$$
$$\|(n+1)!\, P_{n+1}\| \;\leqslant\; \rho^{n+1} \tag{3.42}$$
$$\|(n+1)!\, R_{n+1}\| \;\leqslant\; \rho^{n}\|c\|\;. \tag{3.43}$$

□

The following lemma is the reason why we need a completely irreducible differential module.

Lemma 3.5.2. *Let Y be an open annulus and let $\mathscr{G}$ be a completely irreducible differential equation over Y. Then, for any open or closed sub-annulus $X \subseteq Y$ with $\Gamma_X \subseteq \Gamma_Y$ the connection (cf. (3.17))*

$$\nabla_0 \;:\; \mathrm{End}_0(\mathscr{G}(X)) \xrightarrow{\;\sim\;} \mathrm{End}_0(\mathscr{G}(X)) \tag{3.44}$$

is bijective. Moreover, if X is a closed sub-annulus of Y, one has

$$\|\nabla_0^{-1}\|_X \;<\; +\infty\;. \tag{3.45}$$

Proof. By Lemma 3.4.2, $\mathscr{G}_{|X}$ is completely irreducible. Therefore, the radii of $\mathrm{End}_0(\mathscr{G})$ are log-affine and spectral non solvable along Γ_X. Hence, by Proposition 1.7.1, the connection (3.44) is bijective.

If K is non trivially valued, (3.45) follows from Banach's open mapping theorem (see [Sch02, Corollary 8.7]).

If K is trivially valued, we consider a complete non-trivially valued field extension L/K. By Lemma 3.4.2, $\mathscr{G}_L$ remains completely irreducible over X_L. Indeed, $\mathrm{End}_0(\mathscr{G}(X_L)) = \mathrm{End}_0(\mathscr{G}(X))_L$. Now, we consider the following diagram:

$$\begin{array}{ccc} \mathrm{End}_0(\mathscr{G}(X))_L & \xrightarrow[\sim]{(\nabla_0)_L} & \mathrm{End}_0(\mathscr{G}(X))_L \\ \uparrow & & \uparrow \\ \mathrm{End}_0(\mathscr{G}(X)) & \xrightarrow[\sim]{\nabla_0} & \mathrm{End}_0(\mathscr{G}(X)) \end{array} \tag{3.46}$$

Vertical arrows are injective and isometric by Proposition **??**. Therefore, ∇_0^{-1} is bounded. The claim follows. □

Proposition 3.5.3. *Let $C'' \in \mathscr{C}(C', C)$ and let X be the smallest closed annulus containing C''.*[7] *Then, there exists a constant $\gamma > 0$ (that depends on $\mathscr{F}$ and on X) satisfying the following property: for each $H \in M_r(\mathscr{O}(C))$ such that*

i) $\|H\|_X \leqslant \gamma$,

ii) $\mathrm{tr}(H)$ *has residue zero,*

the differential equations $\mathscr{F}$ and $\mathscr{F}^H$ are isomorphic over C'.

Proof. We denote by ∇ and ∇_0 the connections of End and End_0 respectively. They may act over C, C' or some intermediate annulus, if needed we will specify the space on which they act.

We fix $\rho \in]0, \omega[$ (cf. (0.1)). For all $n \geqslant 1$ we have

$$\frac{\rho^n}{|n!|} < 1 \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{\rho^n}{|n!|} = 0. \tag{3.47}$$

[7]In this proposition it is actually crucial that $X - C'$ is disjoint union of two closed annuli whose skeletons are (both) *not reduced to a point.*

Let $H \in M_r(\mathscr{O}(C))$. The condition expressing that there exists an isomorphism between $\nabla = \frac{d}{dT} + G$ and $\nabla^H = \frac{d}{dT} + G + H$ over C' is the existence of an invertible matrix $M \in GL_r(\mathscr{O}(C'))$ such that

$$\frac{d}{dT} M + GM - M(G+H) \;=\; 0\,, \tag{3.48}$$

that is

$$\nabla_{\mathrm{End}(\mathscr{F}(C'))}(M) \;=\; MH\,. \tag{3.49}$$

We are going to find a matrix M in the form $M = \sum_{n\geqslant 0} M_n$, with $M_n \in M_r(\mathscr{O}(C))$, $M_0 = \mathrm{Id}$ and, for all $n \geqslant 0$,

$$\nabla_{\mathrm{End}(\mathscr{F}(C))}(M_{n+1}) \;=\; M_n H\,. \tag{3.50}$$

Denote by A the smallest closed sub-annulus of C containing C'. We consider the decomposition $\mathrm{End}(\mathscr{F}(C)) = \mathrm{Id}\cdot\mathscr{O}(C) \oplus \mathrm{End}_0(\mathscr{F}(C))$ (cf. (3.18)). By Lemma 3.5.2, the connection ∇_0 (cf. (3.17)) is bijective on $\mathrm{End}_0(\mathscr{F}(C))$ and $\mathrm{End}_0(\mathscr{F}(A))$ and it satisfies

$$\beta \;:=\; \|\nabla_0^{-1}\|_A \;<\; +\infty\,. \tag{3.51}$$

However, the connection of the trivial differential equation $\mathrm{Id}\cdot\mathscr{O}(C)$ is not bijective over closed annuli, it is bijective only over open annuli. For this reason, we need the following

Lemma 3.5.4. *Let U be the image of the connection of $\mathrm{End}(\mathscr{F}(C))$, that is, the space of matrices whose trace has residue zero.*[8] *Let $C'' \in \mathscr{C}(C', C)$ and let X be the smallest closed annulus containing C''. Then, there exists a constant $\alpha > 0$ such that, for each $H \in U$, there exists $M_1 \in \mathrm{End}(\mathscr{F}(C))$ such that*

i) $\nabla(M_1) = H = M_0 H$;

ii) $\|M_1\|_A \leqslant \alpha\|H\|_X$.

Proof. Let $H \in U$. Write $H = H_0 + t_H$ as in (3.4.5). We have just proved that there exists $(M_1)_0 \in \mathrm{End}_0(\mathscr{F}(C))$ such that $\nabla((M_1)_0) = H_0$ and $\|(M_1)_0\|_A \leqslant \beta\|H_0\|_A$. Of course, for each closed annulus X containing A, we also have $\|(M_1)_0\|_A \leqslant \beta\|H_0\|_X$.

Now, the map $f\cdot\mathrm{Id} \mapsto f$ identifies $\mathscr{O}(C)\cdot\mathrm{Id}$ with $\mathscr{O}(C)$ and the connection ∇ corresponds to d/dT (i.e. $\nabla(f\cdot\mathrm{Id}) = f'\cdot\mathrm{Id}$).

Let A and X be as in the statement. If we write $A = \{r_1 \leqslant |T| \leqslant r_2\}$, then $X = \{r_1' \leqslant |T| \leqslant r_2'\}$ with $r_1' < r_1 < r_2 < r_2'$.

Let $f = \sum_{i\in\mathbb{Z}, i\neq -1} a_i T^i \in \mathscr{O}(C)$. We denote by $I(f)$ the primitive of f with zero constant term, *i.e.* $\sum_{k\neq 0} a_{k-1}\frac{T^k}{k}$. We have

$$\|I(f)\|_A = \max\Big(\sup_{k<0}(|a_{k-1}|r_1^k/|k|)\;,\;\sup_{k>0}(|a_{k-1}|r_2^k/|k|)\Big) \tag{3.52}$$

$$\leqslant \max\Big(\sup_{k<0}(|a_{k-1}|{r_1'}^k/|k|)\cdot\sup_{k<0}((r_1/r_1')^k)\;,\;\sup_{k>0}(|a_{k-1}|{r_2'}^k/|k|)\cdot\sup_{k>0}((r_2/r_2')^k)\Big) \tag{3.53}$$

$$\leqslant m\cdot\|f\|_X, \tag{3.54}$$

where m only depends on A and X.

[8] Notice that item (iii) of Lemma 3.4.7 holds because $\mathscr{F}$ is a completely irreducible module and therefore it satisfies the properties (Abs_C) and (L_C) of Section 3.4.2.

Now, set $M_1 := (M_1)_0 + I(t_H)$. Clearly, one has $\nabla(M_1) = H$ and

$$\|M_1\|_A \;\leqslant\; \max\Big(\|(M_1)_0\|_A, \|I(t_H)\|_A\Big) \tag{3.55}$$

$$\leqslant\; \max\Big(\beta\cdot\|H_0\|_A, m\cdot\|t_H\|_X\Big) \tag{3.56}$$

$$\leqslant\; \max(m,\beta)\cdot\max\Big(\|H_0\|_X, \|t_H\|_X\Big)\;. \tag{3.57}$$

To conclude, it is enough to prove that there exists $q > 0$ such that, for each $N \in \mathrm{End}(\mathscr{F}(X))$, we have

$$\max(\|N_0\|_X, \|t_N\|_X) \leqslant q\|N\|_X, \tag{3.58}$$

where $N = N_0 + t_N$ is, as usual, the decomposition in (3.18).

Choose a norm $\|\cdot\|'_X$ on $\mathrm{End}(\mathscr{F}(X))$ coming from a basis adapted to the decomposition $\mathrm{End}(\mathscr{F}(X)) = \mathrm{End}_0(\mathscr{F}(X)) \oplus \mathscr{O}(X)\cdot\mathrm{Id}$. By [Ked10, Lemma 1.3.3], this norm is equivalent to the norm $\|\cdot\|_X$, hence there exists $q_1, q_2 > 0$ such that, for each $N \in \mathrm{End}(\mathscr{F}(X))$, we have

$$q_1\|N\|_X \leqslant \|N\|'_X \leqslant q_2\|N\|_X. \tag{3.59}$$

It follows that, for each $N \in \mathrm{End}(\mathscr{F}(X))$, we have

$$\max(\|N_0\|_X, \|t_N\|_X) \;\leqslant\; 1/q_1\,\max(\|N_0\|'_X, \|t_N\|'_X) \;=\; 1/q_1\,\|N\|'_X \;\leqslant\; q_2/q_1\,\|N\|_X, \tag{3.60}$$

where $\max(\|N_0\|'_X, \|t_N\|'_X) = \|N\|'_X$ by the choice of the basis. The claim follows. □

We come back to the proof of Proposition 3.5.3. We are now ready to initialize our induction. Let us set X as in the statement. Firstly, we set

$$\gamma \;=\; \rho\cdot\min(\alpha^{-1}, \beta^{-1})\;. \tag{3.61}$$

Now, assume that $\mathrm{tr}(H)$ has residue zero and that $\|H\|_X \leqslant \gamma$. We first find a solution to (3.50) for $n = 0$. Since $\mathrm{tr}(H)$ has residue zero, by assumption, there exists $M_1 \in M_r(\mathscr{O}(C))$ such that $\nabla(M_1) = H = M_0H$ and

$$\|M_1\|_A \;\leqslant\; \alpha\cdot\|H\|_X \;\leqslant\; \rho\;. \tag{3.62}$$

We now proceed by induction and assume that we have already constructed $M_0, \ldots, M_n \in \mathrm{End}(\mathscr{O}(C))$ such that, for all $i = 1, \ldots, n$, we have

$$\nabla(M_i) \;=\; M_{i-1}H \tag{3.63}$$

and

$$\|i!\,M_i\|_A \;\leqslant\; \rho^i\;. \tag{3.64}$$

Note that formula (3.37) shows that ∇ defines a derivation on the algebra $\mathrm{End}(\mathscr{F}(C)) \simeq M_r(\mathscr{O}(C))$. By Theorem 3.5.1, for each $n \geqslant 1$, we can write M_nH as

$$M_nH \;=\; \nabla(P_{n+1}) + R_{n+1}\;, \tag{3.65}$$

where R_{n+1} lies in the space generated by the commutators of $\mathrm{End}(\mathscr{F}(C))$ and

$$\|P_{n+1}\|_A \;\leqslant\; \frac{\rho^{n+1}}{|(n+1)!|}\;, \qquad \|R_{n+1}\|_A \;\leqslant\; \frac{\rho^n}{|(n+1)!|}\;. \tag{3.66}$$

In particular, the trace of R_{n+1} is zero (i.e. $R_{n+1} \in \mathrm{End}_0(\mathscr{F}(C))$), hence there exists $Q_{n+1} \in \mathrm{End}_0(\mathscr{F}(C))$ such that

$$\nabla(Q_{n+1}) \;=\; R_{n+1}\;. \tag{3.67}$$

and

$$\|Q_{n+1}\|_A \;\leqslant\; \beta\cdot\|R_{n+1}\|_A. \tag{3.68}$$

The matrix $M_{n+1} = P_{n+1} + Q_{n+1}$ satisfies

$$\nabla(M_{n+1}) \;=\; M_n H \tag{3.69}$$

and Theorem 3.5.1 furnishes the estimation

$$\begin{aligned}
\|M_{n+1}\|_A &\leqslant \max\Big(\|P_{n+1}\|_A, \|Q_{n+1}\|_A\Big) && (3.70)\\
&\leqslant \max\Big(\frac{\rho^{n+1}}{|(n+1)!|}, \beta\|R_{n+1}\|_A\Big) && (3.71)\\
&\leqslant \max\Big(\frac{\rho^{n+1}}{|(n+1)!|}, \beta\frac{\rho^{n}}{|(n+1)!|}\|H\|_A\Big) && (3.72)\\
&\leqslant \frac{\rho^{n+1}}{|(n+1)!|}\;, && (3.73)
\end{aligned}$$

because $\|H\|_A \leqslant \|H\|_X \leqslant \gamma$.

The series $\sum_{n\geqslant 0} M_n$ is therefore convergent in $M_r(\mathscr{O}(A))$ by (3.73). Moreover

$$M \;=\; \mathrm{Id} + \sum_{n\geqslant 1} M_n \tag{3.74}$$

is invertible in $M_n(\mathscr{O}(A))$, because $\|\sum_{n\geqslant 1} M_n\|_A \leqslant \max_{n\geqslant 1}(\rho^n/|n!|) < 1$. The desired equality (3.48) now follows from an easy computation. □

Corollary 3.5.5. *Proposition 3.2.1 holds for a completely irreducible and free differential equation over C.*

Proof. By Corollary 3.4.9 we can assume that $\mathscr{F}$ is completely irreducible. We are therefore in the settings of this subsection. In a basis, $\mathscr{F}$ is given by $\frac{d}{dT} + G$ with $G = \sum_{n\in\mathbb{Z}} G_n \cdot T^n \in M_r(\mathscr{O}(C))$. For all integer $k > 0$ set $G_{[-k,k]} := \sum_{n\notin[-k,k]} G_n T^n$ and $H_k := G - G_{[-k,k]}$. The trace of H_k has a residue equal to zero. Moreover, if X is a closed sub- annulus of C, then for $k \to +\infty$ the norm of $\|H_k\|_X$ goes to zero. Therefore, Proposition 3.5.3 applies and $\mathscr{F}$ is isomorphic to $\frac{d}{dT} + G_{[-k,k]}$. □

Remark 3.5.6. *Let us maintain the notations of Proposition 3.2.1. If the valuation of K is trivial and if $C' = \{r'_1 < |T| < r'_2\}$, with $r'_1 < 1 < r'_2$, then $\mathscr{O}(C') = K[T, T^{-1}]$ (cf. Remark* **??***), hence the claim is trivial. If $r'_1 < r'_2 \leqslant 1$, then $\mathscr{O}(C') = K((T))$ and Proposition 3.2.1 implies the essential surjectivity of Katz's canonical functor [Kat87, Theorem 2.4.10], up to finite scalar extensions of K.*

If the valuation of K is not trivial, Proposition (3.2.1) provides (up to finite scalar extension of K) the essential surjectivity *of the restriction functor associating to a differential module over $K[T, T^{-1}]$ its restriction to $\mathscr{O}(C')$. However, the restriction is not an equivalence as in the Katz's context. For instance, if the residual field of K has positive characteristic p, it is known since Dwork that if π satisfies $\pi^{p-1} = -p$, then the equations $T\frac{d}{dT} - \frac{\pi}{T}$ and $T\frac{d}{dT} - \frac{\pi}{T^p}$ are isomorphic over an annulus of the form $\{r < |T| < 1\}$ for some $r < 1$. This is due to the fact that the exponential $\exp(\pi(T^{-p} - T^{-1}))$ is overconvergent. However, the two equations are not isomorphic over $K[T, T^{-1}]$, since an isomorphism over $K[T, T^{-1}]$ implies an isomorphism over $K((T))$, but this is absurd because they have different irregularities, zero and p, at $T = 0$.*

There is actually an analogous of Katz's canonical extension due to Matsuda [Mat02] and it is no coincidence the fact that a Frobenius structure is required in that case. Indeed, the Frobenius structure implies our condition on the exponents.

3.6. Curves with boundary

In the next section we need to extend the definition of essential algebraizability to all finite curves.

Definition 3.6.1 (Essential algebraizability)**.** *Let X be a finite curve, $Z_0 \subset X$ a finite set of rigid points in X and $\mathcal{F}$ a differential equation on X with meromorphic poles at Z_0. For each boundary point $x \in \partial X$, we consider an elementary tube V_x in $X - Z_0$ centered at x that is adapted to* $\mathrm{End}(\mathcal{F})_{|X-Z_0}$ *(cf. Definition 1.3.1). We set*

$$Y \ := \ X - \bigcup_{x\in\partial X} V_x\ . \tag{3.75}$$

We say that $\mathcal{F}$ is essentially algebraizable if $\mathcal{F}_{|Y}$ is.

Lemma 3.6.2. *The definition is independent of the choices of the V_x's.*

Proof. Let $\{V'_x\}_{x\in\partial X}$ be another choice and set $Y' := X - \bigcup_{x\in\partial X} V'_x$. We have to prove that $\mathcal{F}_{|Y}$ is essentially algebrizable if and only if so is $\mathcal{F}_{|Y'}$. By definition, $V''_x := V_x \cap V'_X$ is again an elementary tube in $X - Z_0$ centered at x that is adapted to $\mathrm{End}(\mathcal{F})_{|X-Z_0}$. Therefore, we have a third curve $Y'' := X - \bigcup_{x\in\partial X} V''_x = Y \cup Y'$ obtained with the same process. Hence, we can replace Y' with Y'' and assume $Y \subseteq Y'$.

The essential algebraizability of $\mathcal{F}_{|Y'}$ clearly implies that of $\mathcal{F}_{|Y}$.

On the other hand, the essential algebraizability of $\mathcal{F}_{|Y}$ implies that of $\mathcal{F}_{|Y'}$. Indeed, $Y' - Y$ is a finite union of virtual open disks D_i on which the radii of $\mathrm{End}(\mathcal{F})$ are constant and on which Corollary 3.1.9 holds. Now, $\mathcal{F}$ being essentially algebrizable on each D_i and on Y, it is essentially algebrizable on the disjoint union $Y \sqcup (\bigsqcup_i D_i) = Y'$. The claim follows. □

3.7. A first application to the finiteness of the cohomology

Lemma 3.7.1. *Let X be a finite curve, Z be a finite set of rigid points in X and $\mathcal{F}$ be a differential equation with meromorphic singularities at Z.*

For each $b \in \partial^o X$, let C_b be an open pseudo-annulus whose skeleton (suitably oriented) represents b. Assume that the C_b's are disjoint and do not meet Z.

For each $b \in \partial^o X$, let C'_b be an open sub-pseudo-annulus of C_b with $\Gamma_{C'_b} \subseteq \Gamma_{C_b}$. Assume that, for all i, the restriction morphism

$$\mathrm{H}^i_{\mathrm{dR}}(C_b, \mathcal{F}_{|C_b}) \xrightarrow{\ \sim\ } \mathrm{H}^i_{\mathrm{dR}}(C'_b, \mathcal{F}_{|C'_b}) \tag{3.76}$$

is an isomorphism,

For each $b \in \partial^o X$, denote by C^∞_b the connected component of $C_b - C'_b$ containing b.[9] *Set $X' := X - \bigsqcup C^\infty_b$. Then, for all i, the restriction morphism*

$$\mathrm{H}^i_{\mathrm{dR}}(X(*Z), \mathcal{F}) \xrightarrow{\ \sim\ } \mathrm{H}^i_{\mathrm{dR}}(X'(*Z), \mathcal{F}) \tag{3.77}$$

is an isomorphism too.

Proof. Set $C := \bigsqcup_b C_b$ and $C' := \bigsqcup_b C'_b$. Then $\{X', C\}$ is an open covering of X. The result now follows from the Mayer-Vietoris exact sequence

$$\cdots \to \mathrm{H}^i_{\mathrm{dR}}(X(*Z), \mathcal{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X'(*Z), \mathcal{F}) \oplus \mathrm{H}^i_{\mathrm{dR}}(C, \mathcal{F}) \to \mathrm{H}^i_{\mathrm{dR}}(C', \mathcal{F}) \to \mathrm{H}^{i+1}_{\mathrm{dR}}(X(*Z), \mathcal{F}) \to \cdots \tag{3.78}$$

□

[9] It may be the empty set.

Remark 3.7.2. *The assumptions of Lemma 3.7.1 are fulfilled for instance in the following cases :*

i) for each b, $\mathcal{F}$ has log-affine radii along Γ_{C_b} and is free of Liouville numbers along it;

ii) for each b, $\mathcal{F}$ has log-affine radii along Γ_{C_b} and it satisfies $(\mathrm{Fin})_{C_b}$.

More general conditions can be expressed using absolute indexes.

Theorem 3.7.3. *Let P be the analytification of a smooth connected projective curve over K. Let Z' be a finite set of rigid points of P. Let $\mathcal{F}$ be a differential equation on P with meromorphic poles at Z'.*

Let X' be an open subset of P that is a finite curve and such that $g(X') = g(P)$. Set $Z := Z' \cap X'$. Assume that, for each $b \in \partial^o X'$, $\mathscr{F}$ has log-affine radii and is free of Liouville numbers along b. Then, the cohomology spaces $\mathrm{H}^\bullet_{\mathrm{dR}}(X'(*Z), \mathcal{F}_{|X'})$ are finite-dimensional and we have

$$\chi_{\mathrm{dR}}(X'(*Z), \mathcal{F}_{|X'}) = \chi_c(X' - Z) \cdot \mathrm{rank}(\mathcal{F}_{|X'}) - \mathrm{Irr}_{X'-Z}(\mathscr{F}) \, . \tag{3.79}$$

Proof. For each $b \in \partial^o X'$, let C'_b be an open pseudo-annulus whose skeleton represents b. Up to restricting it, we may assume that $C'_b \cap Z' = \emptyset$ and that $\mathcal{F}$ has log-affine radii and is free of Liouville numbers along $\Gamma_{C'_b}$. We may assume that the C'_b's are disjoint.

Let $b \in \partial^o X'$. There exists a unique connected component of $P - X'$ that meets the closure of C'_b. It is either a virtual closed disk or a point of type 1 or 4. We denote it by E'_b. The union $D'_b := E'_b \cup C'_b$ is a virtual open disk.

Set $C' := \bigsqcup_b C'_b$ and $D' := \bigsqcup_b D'_b$. Then, $\{X', D'\}$ is a covering of P and we have $X' \cap D' = C'$. Let us consider the Mayer-Vietoris long exact sequence (1.21) for the meromorphic cohomology groups $\mathrm{H}^\bullet_{\mathrm{dR}}(-(*Z'), \mathcal{F})$:

$$\cdots \to \mathrm{H}^i_{\mathrm{dR}}(P(*Z'), \mathcal{F}) \to \mathrm{H}^i_{\mathrm{dR}}(D'(*Z'), \mathcal{F}_{|D'}) \oplus \mathrm{H}^i_{\mathrm{dR}}(X'(*Z'), \mathcal{F}_{|X'}) \to \mathrm{H}^i_{\mathrm{dR}}(C'(*Z'), \mathcal{F}_{|C'}) \to \cdots \tag{3.80}$$

By Corollary ??, the spaces $\mathrm{H}^\bullet_{\mathrm{dR}}(C'(*Z'), \mathcal{F}_{|C'}) = \mathrm{H}^\bullet_{\mathrm{dR}}(C', \mathcal{F}_{|C'})$ are finite dimensional and we have $\chi_{\mathrm{dR}}(C', \mathcal{F}_{|C'}) = 0$. By Corollary ??, the spaces $\mathrm{H}^\bullet_{\mathrm{dR}}(P(*Z'), \mathcal{F})$ are finite dimensional and we have $\chi_{\mathrm{dR}}(P(*Z'), \mathcal{F}) = \chi_c(P - Z') \cdot \mathrm{rank}(\mathcal{F}) - \mathrm{Irr}_{P-Z'}(\mathcal{F})$.

Since P is projective, each D'_b is relatively compact in P. Using that fact that $\mathcal{F}$ is free around the boundary point of D'_b and on each closed sub-disk of D'_b, we deduce that $\mathcal{F}(D'_b)$ is a free $\mathscr{O}(D'_b)[*(Z' \cap D'_b)]$-module. Hence, by Corollary ?? and Theorem ??, the cohomology spaces $\mathrm{H}^\bullet_{\mathrm{dR}}(D'(*Z'), \mathcal{F}_{|D'})$ are finite dimensional too and

$$\chi_{\mathrm{dR}}(D'(*Z'), \mathcal{F}_{|D'}) \; = \; \chi_c(D' - Z') \cdot \mathrm{rank}(\mathcal{F}) - \mathrm{Irr}_{D'-Z'}(\mathcal{F}) + \chi^{\mathrm{abs}}_{b_{D'}}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_{D'}}}) \, , \tag{3.81}$$

where $b_{D'}$ is the germ of segment at the open boundary of D' and $C_{b_{D'}} = C_{b'}$. Again, by Corollary ??, we have $\chi^{\mathrm{abs}}_{b_{D'}}(\mathscr{F}^{\mathrm{Robba}}_{|C_{b_{D'}}}) = 0$.

It follows from (3.80) that the cohomology spaces $\mathrm{H}^\bullet_{\mathrm{dR}}(X'(*Z'), \mathcal{F}_{|X'})$ are finite dimensional too. In addition, the Mayer-Vietoris sequence also gives

$$\chi_{\mathrm{dR}}(X'(*Z'), \mathcal{F}_{|X'}) + \chi_{\mathrm{dR}}(D'(*Z'), \mathcal{F}_{|D'}) - \chi_{\mathrm{dR}}(C', \mathcal{F}_{|C'}) \; = \; \chi_{\mathrm{dR}}(P(*Z'), \mathcal{F}) \, . \tag{3.82}$$

Now, by Proposition 2.1.12, we have $\mathrm{Irr}_{P-Z'}(\mathcal{F}) = \mathrm{Irr}_{X'}(\mathcal{F}_{|X'}) + \mathrm{Irr}_{D'-Z'}(\mathcal{F}_{|D'}) - \mathrm{Irr}_{C'}(\mathscr{F}_{|C'})$ and, by [PP13, Corollary 1.1.55], we have $\chi_c(P - Z') = \chi_c(X' - Z') + \chi_c(D' - Z') - \chi_c(C' - Z')$. The claim follows from the fact that $Z' \cap C' = \emptyset$, which implies that $\chi_c(C' - Z') = \chi_c(C') = 0$, and $\mathrm{Irr}_{C'}(\mathscr{F}_{|C'}) = 0$. □

Corollary 3.7.4. *Let X, P, Z, $\mathcal{F}$ be as in Theorem 3.1.4.*

For each $b \in \partial^o X$, let C_b be an open pseudo-annulus whose skeleton (suitably oriented) represents b. Assume that the C_b's are disjoint and do not meet Z.

For each $b \in \partial^o X$, let C'_b be a relatively compact open sub-pseudo-annulus of C_b with $\Gamma_{C'_b} \subseteq \Gamma_{C_b}$ and denote by C_b^∞ the connected component of $C_b - C'_b$ containing b. Set

$$X' \ := \ X - \bigsqcup C_b^\infty \ . \tag{3.83}$$

For each $b \in \partial^o X$, assume that $\mathcal{F}$ has log-affine radii and is free of Liouville numbers along $\Gamma_{C'_b}$. Then, the cohomology spaces $\mathrm{H}^\bullet_{\mathrm{dR}}(X'(*Z), \mathcal{F}_{|X'})$ are finite-dimensional and (3.79) holds.

Proof. By Theorem 3.1.4, there exists a finite extension L/K such that $(\mathcal{F}_L)_{|X'_L}$ is algebrizable in P_L: there exists a finite subset of rigid points $Z' \subset P_L$ with $Z' \cap X'_L = Z_L$ and a differential equation $\mathcal{F}'$ on P_L with meromorphic poles at Z' such that $\mathcal{F}'_{|X'_L} \cong (\mathcal{F}_L)_{|X'_L}$.

By faithful flatness of L over K, the spaces $\mathrm{H}^\bullet_{\mathrm{dR}}(X'(*Z), \mathcal{F})$ are finite dimensional if, and only if, so are the spaces $\mathrm{H}^\bullet_{\mathrm{dR}}(X'_L(*Z_L), \mathcal{F}_L)$ and, in this case, we have $\chi_{\mathrm{dR}}(X'(*Z), \mathcal{F}) = \chi_{\mathrm{dR}}(X'_L(*Z_L), \mathcal{F}_L)$. Note also that the right hand side of (3.79) is invariant by scalar extension, so it is enough to prove the result for X'_L. Since having log-affine radii and being free of Liouville numbers is preserved by scalar extension, the result now follows from Theorem 3.7.3. □

Remark 3.7.5. *Notice that:*

i) No Liouville condition are required at the germs of segments out of points of Z;

ii) In (3.79) one has $\chi_c(X' - Z) = \chi_c(X - Z)$ and that $\mathrm{Irr}_{X'-Z}(\mathcal{F}) = \mathrm{Irr}_{X-Z}(\mathcal{F})$ by affineness of the radii of $\mathcal{F}$ along Γ_{C_k}.

Remark 3.7.6. *There is a generalized version of Theorem 3.1.4 and Corollary 3.7.4 where one allows boundary to X and one proves that $\mathrm{H}^\bullet_{\mathrm{dR}}(P(*Z), \mathcal{F})$ is finite dimensional if and only if so is $\mathrm{H}^\bullet_{\mathrm{dR}}(X'(*Z), \mathscr{F})$. Now, in this paper we systematically exclude the boundary by assuming that the radii of the equation are spectral non solvable there in order to apply Corollary 1.7.3. Equations with solvable radii at the boundary are not investigated, although in some cases such equations may have finite dimensional cohomology. The generalization of Corollary 3.7.4 may be actually useful in that context. However, the statement becomes more technical and we prefer the simplified version.*

4. Index of differential equations over finite curves

In this section, we apply the material we have previously developed to prove some index results for differential equations over finite curves.

The class of differential equations considered here is the family of differential equations $\mathcal{F}$ over a *finite curve* X with a finite number of *meromorphic singularities*, having *spectral non-solvable radii* at the points of the boundary ∂X of X and that are *strongly free of Liouville numbers* at the germs of segments at the open boundary $\partial^o X$ of X (cf. Definition **??**). Notice that we do not necessarily require log-affinity of the radii at the open boundary.

For this class, we obtain necessary and sufficient conditions for the *finite dimensionality of the de Rham cohomology* together with *index formulas*. We provide two criteria relating the finiteness of the cohomology of the equation to the behavior of its *radii of convergence functions* (cf. Theorem 4.1.1).

More specifically, the first criterion asserts that the cohomology is finite dimensional if, and only if, the total height of the convergence Newton polygon of the equation is a *log-affine function* at every germ of segment at the open boundary of X. In this case, we obtain an index formula in the Grothendieck-Ogg-Shafarevich style (cf. Formula (4.3)), where the index of the equation equals the characteristic of the curve multiplied by the rank of the equation minus the *global irregularity* defined in Section 2.

Together with this description, we also derive another criterion asserting that, when the skeleton Γ_S is a finite union of intervals, the cohomology is finite dimensional if, and only if, *the virtual local indexes* $\chi(x,\Gamma_S,\mathscr{F})$ *vanish* at almost all points x of S. In this case, we obtain a second index formula (cf. Formula (4.29)) expressing the index as the sum of the virtual local indexes $\chi(x,\Gamma_S,\mathscr{F})$. This second criterion will be relevant in Section 5, where we consider general quasi-smooth K-analytic curves.

We also provide some comparison results between *meromorphic* and *analytic* cohomologies under further non-Liouville conditions at the meromorphic singularities (cf. Corollary **??**). This comparison is interesting because it asserts in particular that *any* analytic solution of the equation is actually meromorphic around a meromorphic singularity (cf. Remark 4.1.3).

The affinity assumption on the radii required by the first criterion arises automatically in three situations:

i) around a meromorphic singularity (cf. Lemma **??**);

ii) in a relatively compact situation (cf. Section 4.2);

iii) if the equation is overconvergent (cf. Section 4.3).

If the curve is compact or affinoid, the overconvergent case can be seen as a particular case of the relatively compact case (cf. Remark 4.3.3). Moreover, in the overconvergent case our finiteness results hold without the assumption about the spectral non-solvability of the radii of $\mathscr{F}$ at the boundary of X. Indeed, the theory of overconvergent differential equations behaves as if X had no boundary (cf. Remark 4.3.1).

4.1. Differential equations on finite curves

In this section we establish a first index result for finite curves. Recall that we provided criteria characterizing finite curves in [PP13, Section 1.1.9].

Let X be a finite curve, $Z \subseteq X$ a finite set of rigid points in X and $\mathcal{F}$ a differential equation on X with meromorphic singularities at Z. We set

$$Y := X - Z \quad \text{and} \quad \mathscr{F} := \mathcal{F}_{|Y} . \tag{4.1}$$

Clearly, Y is a finite curve too and $\partial X = \partial Y$. Moreover $\partial^o Y = \partial^o X \cup \{b_z \ , \ z \in Z\}$, where, for each $z \in Z$, b_z is the germ of segment out of the point z.

For each point $x \in Y$ we have a maximal tube $V_{\mathrm{m}}(x,\mathscr{F})$ centered at x and a set of singular directions $\mathrm{Sing}_m(x,\mathscr{F})$ out of x (cf. Definition 1.3.12). We set

$$\mathrm{Sing}_m(\partial Y,\mathscr{F}) := \bigcup_{x\in\partial Y} \mathrm{Sing}_m(x,\mathscr{F}) . \tag{4.2}$$

We will use below the notions of virtual local indexes $\chi(x,\Gamma_S,\mathscr{F})$ (cf. Definition 2.2.1 and Section 2.2.4), of equation free of Liouville numbers at a a germ of segment (cf. Definition **??** and Section **??**) and of quasi-finite graph (cf. [PP13, Definition 1.1.32]).

The following theorem is our main result.

Theorem 4.1.1. *Let X, Z, Y, $\mathcal{F}$, $\mathscr{F}$ as above.*

Assume that

i) for each $x \in \partial Y$, the radii of $\mathscr{F}$ are all spectral non-solvable at x;

ii) for each $b \in \mathrm{Sing}_m(\partial Y, \mathscr{F}) \cup \partial^o X$, $\mathscr{F}$ is strongly free of Liouville numbers along b.[10]

Then, the following conditions are equivalent:

a) for each $i \geqslant 0$, the meromorphic de Rham cohomology space $\mathrm{H}^i_{\mathrm{dR}}(X(*Z), \mathcal{F})$ is finite-dimensional;

b) $\mathcal{F}$ has well-defined irregularity[11].

Moreover, when these properties hold, we have the index formula

$$\chi_{\mathrm{dR}}(X(*Z), \mathcal{F}) \;=\; \sum_{i=1}^{n}\Big(\chi_c(Y_i)\cdot \mathrm{rank}(\mathscr{F}_{|Y_i})\Big) - \mathrm{Irr}_Y(\mathscr{F})\;, \tag{4.3}$$

where $Y_1, \ldots, Y_n$ are the connected components of Y.[12]

Let S be a pseudo-triangulation of Y such that Γ_S is quasi-finite and meets every connected component of Y.[13] *Then a) and b) are also equivalent to*

c) there exists a finite subset F of Γ_S such that, for each $x \in \Gamma_S - F$, one has

$$\chi(x, \Gamma_S, \mathscr{F}) \;=\; 0\,. \tag{4.4}$$

When it holds, we have

$$\chi_{\mathrm{dR}}(X(*Z), \mathcal{F}) \;=\; \sum_{x\in\Gamma_S} \chi(x, \Gamma_S, \mathscr{F})\,. \tag{4.5}$$

If, moreover, S is adapted to $\mathscr{F}$ (cf. Definition 2.2.5), the previous conditions are also equivalent to

d) there exists a finite subset F of S such that, for each $x \in S - F$, one has

$$\chi(x, \Gamma_S, \mathscr{F}) \;=\; 0\,. \tag{4.6}$$

When it holds, we have

$$\chi_{\mathrm{dR}}(X(*Z), \mathcal{F}) \;=\; \sum_{x\in S} \chi(x, \Gamma_S, \mathscr{F})\,. \tag{4.7}$$

Proof. We split our proof into several steps.

Equivalence between b), c) and d). The points b), c) and d) actually concern $\mathscr{F}$ and Y. Therefore, their equivalence follows from Proposition 2.2.8. Hence, we only have to prove the equivalence between (a) and (b) and that, when they are satisfied, formula (4.3) holds.

Reduction to the case where X is connected and without boundary. All the quantities being additive on the connected components of X, we can assume X connected. In this case, the rank of $\mathscr{F}$ is a constant function on Y and the right hand side of (4.3) then becomes $\chi_c(Y)\cdot \mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_Y(\mathscr{F})$. Set $r := \mathrm{rank}(\mathscr{F})$.

Let S be a pseudo-triangulation of Y. By definition, for each $x \in \partial Y$, the connected components of $V_{\mathrm{m}}(x, \mathscr{F}) - \{x\}$ are virtual open disks. As a consequence, the set

$$S_{\mathrm{m}} := S - \bigcup_{x\in\partial Y}\big(V_{\mathrm{m}}(x, \mathscr{F}) - \{x\}\big) \tag{4.8}$$

is still a pseudo-triangulation of Y. From now on, we fix this pseudo-triangulation and compute the radii of convergence of $\mathscr{F}$ with respect to it. Again by definition, for each $x \in \partial Y$, $V_{\mathrm{m}}(x, \mathscr{F})$ is an

[10] Notice that no Liouville condition are required at the germs of segments out of the points of Z.
[11] see Section 2.1.1. Note that this condition is always satisfied if K is trivially valued
[12] If $X_1, \ldots, X_m$ are the connected components of X, we have $m = n$ and $Y_i = X_i - Z$ for all i.
[13] We do not assume that S is also a pseudo-triangulation of X (cf. Remark 2.2.6).

elementary tube centered at x that is adapted to $\mathscr{F}$ (cf. Definition 1.3.1). Recall that $\partial X = \partial Y$ and set

$$X_0 \ := \ X - \bigcup_{x\in\partial X} V_{\mathrm{m}}(x,\mathscr{F}) \quad \text{and} \quad Y_0 := Y - \bigcup_{x\in\partial X} V_{\mathrm{m}}(x,\mathscr{F})\ . \tag{4.9}$$

We have $Y_0 = X_0 - Z$.

The right hand side of the index formula (4.3) only involves $\mathscr{F}$ and Y. Therefore, we are allowed to apply Proposition 2.1.14: $\mathscr{F}_{|Y_0}$ has well-defined irregularity if, and only if, so has $\mathscr{F}$ and, in this case, we have

$$\chi_c(Y)\cdot r - \mathrm{Irr}_Y(\mathscr{F}) \ = \ \chi_c(Y_0)\cdot r - \mathrm{Irr}_Y(\mathscr{F}_{|Y_0})\ . \tag{4.10}$$

On the other hand, by Corollary 1.7.3, we have a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) \ \xrightarrow{\ \sim\ } \ \mathrm{H}^i_{\mathrm{dR}}(X_0(*Z),\mathcal{F}_{|X_0})\ . \tag{4.11}$$

Now, by Remark 1.7.4, X_0 is a finite disjoint union of connected finite curves without boundary that satisfy the conditions of the statement. We may now assume that X is connected and without boundary.

Equivalence between a) and b) for X connected and without boundary. If X is projective, the claim follows from Proposition 1.5.17.

Let us now assume that X is not projective. By Theorem 1.2.2, it is cohomologically Stein. For each $b\in\partial^o X$, let C_b be an open pseudo-annulus in X whose skeleton (suitably oriented) represents b. We may assume that the C_b's are disjoint and that $Z\cap C_b = \emptyset$.

By assumption ii), $\mathscr{F}$ is free of Liouville numbers at every germ of segment at the open boundary of X. This ensures that, for each $b\in\partial^o X$, we can find an open relatively compact sub-annulus C'_b of C_b such that $\mathcal{F}_{|C'_b}$ has log-affine radii along $\Gamma_{C'_b}$ and is free of Liouville numbers along it.

The complement $C_b - C'_b$ has two connected components. We call C_b^∞ the one that contains b. Set

$$C^\infty \ := \ \bigcup_{b\in\partial^o X} C_b^\infty\ . \tag{4.12}$$

The curve $X' := X - C^\infty$ is open, relatively compact in X and each element of its open boundary is represented by the skeleton (suitably oriented) of one and only one element of the family $\{C'_b\}_b$. Moreover, $\mathscr{F}$ is free of Liouville numbers at its open boundary.

Now, again by assumption ii), $\mathrm{End}(\mathscr{F})$ is also free of Liouville numbers at every germ of segment at the open boundary of X. Therefore, we can construct in a similar way another open relatively compact subset X^e of X such that $\mathrm{End}(\mathscr{F})$ is free of Liouville numbers at its open boundary. Moreover, we may construct it in such a way that X^e contains X' as an open and relatively compact subset.

By Lemma 3.1.2, X^e admits an open embedding into a projective curve P such that $P - X^e$ is a disjoint union of virtual closed disks. The equation $\mathrm{End}(\mathcal{F}_{|X^e})$ is free of Liouville numbers at the open boundary of X^e. All the assumptions of Corollary 3.7.4 are now satisfied, with X replaced by X^e.

It follows that the cohomology spaces $\mathrm{H}^i_{\mathrm{dR}}(X'(*Z),\mathcal{F}_{|X'})$ are finite dimensional and that we have

$$\chi_{\mathrm{dR}}(X'(*Z),\mathcal{F}_{|X'}) \ = \ \chi_c(Y')\cdot \mathrm{rank}(\mathcal{F}) - \mathrm{Irr}_{Y'}(\mathcal{F}), \tag{4.13}$$

where $Y' := X' - Z$. Notice that $\chi_c(Y') = \chi_c(Y)$ because Y' is obtained from Y by shrinking the open boundary.

For each b, we now consider $C_b^+ := C_b' \cup C_b^\infty$ and set

$$C^+ \;:=\; \bigcup_b C_b^+ \quad \text{and} \quad C' \;:=\; \bigcup_b C_b' \,. \tag{4.14}$$

Each C_b^+ is an open pseudo-annulus at the open boundary of X and we have $C^+ \cap X' = C'$. We then consider the Mayer-Vietoris long exact sequence for the open covering $\{X', C^+\}$ of X:

$$\cdots \to \mathrm{H}^{i-1}_{\mathrm{dR}}(C',\mathcal{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X'(*Z),\mathcal{F}) \oplus \mathrm{H}^i_{\mathrm{dR}}(C^+,\mathcal{F}) \to \mathrm{H}^i_{\mathrm{dR}}(C',\mathcal{F}) \to \cdots \tag{4.15}$$

By Corollary **??**, the cohomology spaces $\mathrm{H}^i_{\mathrm{dR}}(C',\mathcal{F}_{|C'})$ are finite dimensional and we have $\chi_{\mathrm{dR}}(C',\mathcal{F}_{|C'}) = 0$. It follows that the spaces $\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F})$ are finite dimensional if, and only if, so are the spaces

$$\mathrm{H}^i_{\mathrm{dR}}(C^+,\mathcal{F}) \;=\; \bigoplus_{b\in\partial^o X} \mathrm{H}^i_{\mathrm{dR}}(C_b^+,\mathcal{F}) \,. \tag{4.16}$$

By Corollary **??**, for each $b \in \partial^o X$, the spaces $\mathrm{H}^i_{\mathrm{dR}}(C_b^+,\mathcal{F}) = \mathrm{H}^i_{\mathrm{dR}}(C_b^+,\mathscr{F})$ are finite dimensional if, and only if, $\mathscr{F}_{|C_b^+}$ has well-defined irregularity.

The equivalence between a) and b) follows.

Proof of (4.3). Assume now that a) and b) hold. This implies that, for each $b \in \partial^o X$, by choosing C_b close enough to the boundary of X, we may assume that $\mathrm{Irr}_{C_b}(\mathscr{F}) = \mathrm{Irr}_{C_b'}(\mathscr{F}) = 0$. By Theorem 1.4.9, it follows that the restrictions

$$\mathrm{H}^i_{\mathrm{dR}}(C_b,\mathscr{F}) \xrightarrow{\;\sim\;} \mathrm{H}^i_{\mathrm{dR}}(C_b',\mathscr{F}) \,, \tag{4.17}$$

are isomorphisms for all i. Hence, by Lemma 3.7.1, the restrictions $\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X'(*Z),\mathcal{F})$ are isomorphisms for all i too, and in particular $\chi_{\mathrm{dR}}(X(*Z),\mathcal{F}) = \chi_{\mathrm{dR}}(X'(*Z),\mathcal{F})$.

On the other hand, the fact that $\mathscr{F}$ has well-defined irregularity implies that $\mathrm{Irr}_Y(\mathscr{F}) = \mathrm{Irr}_{Y'}(\mathscr{F})$. Formula (4.3) follows. □

Corollary 4.1.2. *Let X be a finite curve with no boundary. Then, for each $i \geqslant 0$, the de Rham cohomology space $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{O})$ of the trivial differential equation is finite-dimensional. Moreover, we have*

$$\chi_{\mathrm{dR}}(X,\mathscr{O}) \;=\; \chi_c(X) \,. \tag{4.18}$$

Remark 4.1.3. *By Corollary* **??**, *in presence of further conditions at the germs of segments out of the points of Z, we also have isomorphisms between meromorphic and analytic cohomologies:*

$$\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X - Z,\mathcal{F}_{|X-Z}) \,, \quad \forall\, i \,. \tag{4.19}$$

We notice that this implies, in particular, the following interesting property :

$$\textit{Every analytic solution of } \mathcal{F} \textit{ on } Y \textit{ is meromorphic at } Z. \tag{4.20}$$

This generalizes a similar principle obtained in [Bal87] for projective curves.

Conjecture 4.1.4. *For the purely analytic cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$, descent is done in Appendix* **??** *and it requires a quite fine set of results involving in a decisive way the assumption that the spaces are Fréchet.*

For the meromorphic cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F})$, we are unable to prove descent by arbitrary extension of the ground field K. Of course, Remark 4.1.3 shows that with further Liouville conditions at Z there is no distinction between analytic and meromorphic cohomologies, and in this case we have arbitrary descent by the results of Appendix **??**. *Also, if X is projective, by*

Proposition 1.5.10 and Corollary 1.5.9, we may reduce to a purely algebraic setting and descent then holds since field extensions are faithfully flat. However, the general case remains open.

The index result provided by Theorem 4.1.1 shows that the index $\chi_{\mathrm{dR}}(X(*Z),\mathcal{F})$ is invariant by scalar extension without Liouville assumptions at the points of Z. It seems then natural to conjecture that the cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(X(*Z),\mathcal{F})$ are themselves invariant by scalar extension of K.

Remark 4.1.5. *We here discuss the assumptions of Theorem 4.1.1.*

- *i) Assumption i) is about the values of the radii at the boundary points of X. The cohomology of the trivial equation over a closed disk or annulus has an infinite dimensional cohomology, therefore we may not expect to suppress assumption i) without further finer assumptions.*
- *ii) Assumption ii) is more delicate. We believe that this hypothesis can be generalized, but at the present state of the theory the Liouville assumption on the exponents seems unavoidable in order to obtain the essential algebraicity of the equation (cf. Section 3). This belief is supported by the fact that in several situations, over the affine line, the exact conditions providing the finite dimensionality of the cohomology are not directly related to the exponents (cf. for instance Theorems* **??** *and* **??***).*
- *iii) Condition (c) of Theorem 4.1.1 requires Γ_S to be quasi-finite or, equivalently, that the set of end-points of Γ_S is finite. By Remark 2.2.9, we know that this condition is necessary for the equality (2.53). We will encounter similar situations in Section 5 in the case of general curves.*
- *iv) The case where Z is locally finite but possibly not finite will not be discussed in this paper. Indeed, if Z is locally finite but not finite, the curve Y is not finite anymore and we are in the setting of Section 5. For similar reasons as those of Conjecture 4.1.4 we are unable to extend the Christol-Mebkhout limit process (cf. Section* **??***) to the meromorphic case.*

The finite dimensionality of the de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ does not imply the quasi-finiteness of the controlling graph $\Gamma_S(\mathscr{F})$. Below we give an explicit example of a differential equation with finite de Rham cohomology, but infinite controlling graph.

Example 4.1.6. *Let $D := \{|T| < 1\}$ be the open unit disk and, for all $\rho < 1$, let x_ρ be the sup-norm on the sub-disk $\{|T| \leqslant \rho\}$. Let $\mathscr{F}$ be the differential module on the open unit disk D defined in a cyclic basis by the operator $(\frac{d}{dT})^2 + f_1(\frac{d}{dT}) + f_2$, where*

- *i) f_1 is a bounded function on D with infinitely many zeros;*
- *ii) $f_2 \in K$ is a constant function whose norm satisfies $|f_2| > 1$;*
- *iii) the sup-norm $\|f_1\|_D := \sup_{x\in D} |f_1|(x)$ satisfies*

$$1 \;<\; \|f_1\|_D \;<\; |f_2| \;<\; \|f_1\|_D^2\,. \tag{4.21}$$

By (4.21), there exists $s < 1$, such that for all $\rho \in]s,1[$ one has

$$\rho^{-1} \;<\; |f_1|(x_\rho) \;<\; \rho\cdot|f_2| \;<\; \rho\cdot|f_1|(x_\rho)^2\,. \tag{4.22}$$

By Young's Theorem [You92] (see [Pul15, Section 4.3] for notations close to our setting), for all $\rho \in]s,1[$ one has

$$\mathcal{R}_{\emptyset,1}(x_\rho,\mathscr{F}) \;=\; \frac{\omega}{|f_1|(x_\rho)} \;<\; \omega\rho \tag{4.23}$$

$$\mathcal{R}_{\emptyset,2}(x_\rho,\mathscr{F}) \;=\; \frac{\omega\cdot|f_1|(x_\rho)}{|f_2|} \;<\; \omega\rho\,, \tag{4.24}$$

where ω is defined in (0.1). The total height of the convergence Newton polygon of $\mathscr{F}$ is constant *on the interval* $I_s := \{x_\rho \mid s < \rho < 1\}$ *with value*

$$H_{\emptyset,2}(x_\rho, \mathscr{F}) \;=\; \frac{\omega^2}{|f_2|}\,. \tag{4.25}$$

In particular, it is log-affine and $\mathscr{F}$ *has well-defined irregularity.*

Denote by b_D *the gem of segment at the open boundary of* D. *Since the radii of* $\mathscr{F}$ *are spectral non-solvable along* b_D, *by Proposition* **??** *and Corollary* **??**, *we have*

$$h^1(D, \mathscr{F}) = \partial_{b_D} H_{\emptyset,2}(x_\rho, \mathscr{F}) = 0. \tag{4.26}$$

Notice that $i = 1$ *is a vertex of the convergence Newton polygon for all* $x \in I_s$, *i.e.* $i = 2$ *separates the radii along* I_s. *By Theorem 2.1.1, the first radius is hence* harmonic *over* I_s, *so* $\Gamma_{\emptyset,1}(\mathscr{F})$ *has a bifurcation at each point of* I_s *where* $\mathcal{R}_{\emptyset,1}(-, \mathscr{F})$ *has a break. Since* f_1 *has infinitely many zeros,* $\Gamma_{\emptyset,1}(\mathscr{F})$ *has infinitely many bifurcation points along* I_s.

Since $H_{\emptyset,2}(-, \mathscr{F})$ *is constant along* I_s, $\Gamma_{\emptyset,2}(\mathscr{F})$ *has the same bifurcation points as* $\Gamma_{\emptyset,1}(\mathscr{F})$ *along it. This shows that* $\Gamma_S(\mathscr{F})$ *is not quasi-finite.*

Besides, we notice that this is an example of a differential equation which is indecomposable over D *and also over any annulus at the boundary of* D. *Indeed, the second radius function* $\mathcal{R}_{\emptyset,2}(-, \mathscr{F})$ *is a convex function along* I_s *and cannot be the first radius* $\mathcal{R}_{\emptyset,1}(-, \mathscr{F}_1)$ *of a rank one sub-differential module* $\mathscr{F}_1 \subset \mathscr{F}$.

We now provide an example of a differential equation satisfying items i) and ii) of Theorem 4.1.1, with infinite dimensional de Rham cohomology.

Example 4.1.7. *Let* $D = \{|T| < r\}$ *be an open disk, and let* x_ρ *be as in the above example. Let* $f(T)$ *be any* unbounded *analytic function on* D *and let* $\mathscr{F}$ *be the differential equation on* D *associated with the operator* $\frac{d}{dT} - f(T)$. *By Proposition* **??**, *for all* $\rho < r$ *close enough to* r, *we have* $\mathcal{R}_{\emptyset,1}(x_\rho, \mathscr{F}) = \frac{\omega}{r \cdot |f(x_\rho)|} < \omega\rho/r$. *It follows that* $\mathscr{F}$ *has spectral non-solvable radii at the open boundary of* D *and that it is hence free of Liouville numbers at the open boundary of* D. *Moreover,* $\mathscr{F}$ *does not have well defined irregularity since the total height of its convergence Newton polygon coincides with* $\mathcal{R}_{\emptyset,1}(-, \mathscr{F})$. *Therefore, Theorem 4.1.1* applies *and* $\dim \mathrm{H}^1_{\mathrm{dR}}(D, \mathscr{F}) = +\infty$.

4.2. Relatively compact curves

Two of the most important properties required by Theorem 4.1.1 in order to have finite dimensional cohomology are the finiteness of the curve X and the fact that $\mathscr{F}$ has well-defined irregularity on it. In this section, we show that these conditions are automatic in a relatively compact situation.

Concerning the finiteness of X, we already provided several statements showing that when X is conveniently embedded into another curve Y, then X is automatically finite (cf. [PP13, Lemmas 1.1.35, 1.1.36, 1.1.56, and Remark 1.1.57]).

On the other hand, if we assume that $\mathcal{F}$ extends to a differential equation $\mathcal{F}'$ on Y, then its radii extend too and since they are locally finite functions (cf. [Pul15, PP15]), they all have a finite number of breaks at the open boundary of X, hence $\mathcal{F}$ automatically has well-defined irregularity. We resume these considerations in the following statement.

Proposition 4.2.1. *Let* X *and* Y *be quasi-smooth* K*-analytic curves such that* X *is embedded into* Y *as an open analytic domain. Let* Z *be a finite set of rigid points of* X *and let* $\mathcal{F}'$ *be a differential equation on* Y *with meromorphic singularities at* Z. *Denote by* $\mathcal{F}$ *the restriction of* $\mathcal{F}'$ *to* X. *Assume that:*

i) X *has finitely many connected components;*

ii) X has finite genus;

iii) X has finite boundary ∂X;

iv) there exists a pseudo-triangulation S of X such that Γ_S is quasi-finite;

v) Γ_S is relatively compact in Y;

Then, X is a finite curve and $\mathcal{F}$ has well-defined irregularity on X.

In particular, if the assumptions i) and ii) of Theorem 4.1.1 are fulfilled, then $\mathcal{F}$ has finite dimensional cohomology and (4.3) holds, as well as the remaining part of the statement. □

4.3. Overconvergent differential equations

An important class of equations for which relative compactness plays a crucial role is that of overconvergent differential equations over compact curves (cf. Remark **??**). However, we place ourself in a more general situation and do not assume X compact (*i.e.* we allow some open boundary $\partial^o X$ in X) nor that X is relatively compact in X'.

Let X be a finite curve, Z be a finite set of rigid points in X, $\mathcal{F}$ be an overconvergent differential equation with meromorphic singularities at Z. Recall that this means that X embeds as an analytic domain into a smooth K-analytic curve X' (with no boundary) and there exists a connection $(\mathcal{F}',\nabla')$ on X' with meromorphic singularities on Z whose restriction to X is $(\mathcal{F},\nabla)$. As usual, we set $Y := X - Z$, $Y' := X' - Z$, $\mathscr{F} := \mathcal{F}_{|Y}$ and $\mathscr{F}' := \mathcal{F}'_{|Y'}$. In this case, $\mathscr{F}$ is a differential equation on Y which is overconvergent in Y'.

Recall that we have already defined the overconvergent meromorphic de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(X^\dagger(*Z),\mathcal{F})$ (cf. Definition 1.6.8), the overconvergent Euler characteristic $\chi_c(X^\dagger)$ (cf. Definition 1.3.8), the overconvergent Liouville conditions (cf. Definition 1.6.9), the overconvergent irregularity $\mathrm{Irr}_{Y^\dagger}(\mathscr{F})$ (cf. Definition 2.1.20), the overconvergent virtual local indexes (cf. Definition 2.2.12) and the fact that $\mathcal{F}$ has well-defined irregularity (cf. Definition 2.1.19).

Remark 4.3.1. *We remind that all these invariants are actually associated to $\mathcal{F}'_{|U}$ for some unspecified elementary neighborhood U of X in X' that is adapted to $\mathcal{F}$ (cf. Definition 2.1.17).*

Observe also that we do not assume that the radii of $\mathcal{F}$ are spectral non-solvable at the points of the boundary ∂X. Indeed, those points now have to be considered as internal points of U. In other words the overconvergent theory behaves as if there were no boundary in X.

Corollary 4.3.2. *We maintain the notations above. Let S be a pseudo-triangulation of Y. Assume that*

i) for each $b \in \partial^o X$, $\mathscr{F}$ is strongly free of Liouville numbers along b.[14]

ii) $\mathscr{F}$ is strongly free of overconvergent Liouville numbers at every point x of the boundary ∂X.

Then, the following conditions are equivalent:

(a) for each $i \geqslant 0$, the overconvergent meromorphic de Rham cohomology group $\mathrm{H}^i_{\mathrm{dR}}(X^\dagger(*Z),\mathcal{F})$ is finite-dimensional;

(b) $\mathcal{F}$ has well-defined irregularity (cf. Definition 2.1.19).

Moreover, when these properties hold, we have the index formula

$$\chi_{\mathrm{dR}}(X^\dagger(*Z),\mathcal{F}) \;=\; \sum_{i=1}^{n}\Big(\chi_c^\dagger(Y_i)\cdot \mathrm{rank}(\mathscr{F}_{|Y_i})\Big) - \mathrm{Irr}_Y^\dagger(\mathscr{F})\;, \tag{4.27}$$

[14] Again, no Liouville conditions are required at the germs of segments out of points of Z.

where $Y_1, \dots, Y_n$ are the connected components of Y.[15]

Assume now that S is a pseudo-triangulation of Y adapted to $\mathscr{F}$ (cf. Definition 2.2.5) that meets all connected components of Y and such that $\Gamma_S \subseteq Y$ is quasi-finite. Then (a) and (b) are also equivalent to

(c) there exists a finite subset S' of S such that, for each $x \in S - S'$, one has

$$\chi^\dagger(x, \Gamma_S, \mathscr{F}) \;=\; 0\,. \tag{4.28}$$

In this case, we also have the index formula

$$\chi_{\mathrm{dR}}(X^\dagger(*Z), \mathcal{F}) \;=\; \sum_{x\in S} \chi^\dagger(x, \Gamma_S, \mathscr{F})\,. \tag{4.29}$$

Proof. Let U be an elementary neighborhood of X in X' that is adapted to $\mathcal{F}$ (cf. Definition 2.1.17). By Lemma 1.6.11, for all i we have a natural isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(U(*Z), \mathcal{F}) \;\xrightarrow{\sim}\; \mathrm{H}^i_{\mathrm{dR}}(X^\dagger(*Z), \mathcal{F})\,. \tag{4.30}$$

The result then follows from Theorem 4.1.1 and Remark 4.3.1. □

Remark 4.3.3. *As evoked at the beginning of this section, if X is compact, then, by finiteness of the radii (cf. [Pul15, PP15]), $\mathcal{F}$ automatically has well-defined irregularity.*

In this case, we can deduce Corollary 4.3.2 by the following argument. Firstly we use (4.30) to pass from the cohomology over $X^\dagger(*Z)$ to that over a suitable elementary neighborhood $U(*Z)$ of X in X' that is adapted to $\mathcal{F}$. Now, if $U' \subset U$ is another elementary neighborhood of X which is relatively compact in U, then we have again isomorphisms $\mathrm{H}^i_{\mathrm{dR}}(U(*Z), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X^\dagger(*Z), \mathcal{F})$, for all i. We then use Proposition 4.2.1 to deduce the finite dimensionality.

5. Index of differential equations over arbitrary curves.

In this section we deal with differential equation over general curves. The basic idea is to express our curve X as a limit of certain open relatively compact sub-curves X_n and then use the Christol-Mebkhout limit process of Section **??**. The class of differential equations that is taken onto account is described by some conditions that are essentially those of Theorem 4.1.1 on every X_n. For this class we obtain a necessary and sufficient criterion for the finite-dimensionality of the de Rham cohomology which is based on a interpretation of the index as a sum of the local contributions $\chi(x, \Gamma_S, \mathscr{F})$ (cf. (4.4)), the condition corresponds to the vanishing of these virtual local indexes for all but a finite number of points x.

Here we will only consider the analytic cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(X, \mathscr{F})$ with no meromorphic singularities. The reason is that we are not able to extend the limit formula of Section **??** outside this framework.

5.1. A first criterion for finite-dimensionality of de Rham cohomology.

Let X be a quasi-smooth K-analytic curve and let $\mathscr{F}$ be a differential equation over X. If X is connected and compact, then, by [Duc, Théorème 6.1.3], it is either affinoid (if the boundary is not empty) or projective. In both cases it is a finite curve, hence taken into account by Theorem 4.1.1.

Here we are interested in the case where X may fail to be finite. We want to use Christol-Mebkhout's formula (see Theorem 1.4.9) in order to compute the de Rham cohomology of $\mathscr{F}$ on X.

[15] If $X_1, \dots, X_m$ are the connected components of X, we have $m = n$ and $Y_i = X_i - Z$ for all i. Recall also that $\chi_c(Y_i^\dagger) = \chi_c(X_i^\dagger) - \sum_{z\in Z\cap X_i}[\mathscr{H}(z) : K]$, where $\mathscr{H}(z)$ is the residue field of z.

It requires writing the curve X as an increasing union of curves X_n, for $n \geqslant 0$, with finite-dimensional de Rham cohomology. When the curves X_n are finite, Theorem 4.1.1 provides sufficient conditions to ensure this.

Theorem 5.1.1. *Assume that X is connected and not proper. Assume moreover that there exists a non-decreasing sequence of analytic domains $(X_n)_{n\in\mathbb{N}}$ of X forming a covering of X for the G-topology and an integer n_0 such that, for every $n \geqslant n_0$,*

i) X_n is a connected finite curve;

ii) the restriction map $\mathscr{O}(X_{n+1}) \to \mathscr{O}(X_n)$ has dense image;

iii) for each $x \in \partial X_n$, the radii of $\mathscr{F}$ are all spectral non-solvable at x;

iv) for each $b \in \mathrm{Sing}_m(\partial X_n, \mathscr{F}_{|X_n})$, $\mathscr{F}_{|X_n}$ is strongly free of Liouville numbers along b;

v) for each $b \in \partial^o X_n$, the total height of the Newton polygon of $\mathscr{F}_{|X_n}$ is log-affine along b and $\mathscr{F}_{|X_n}$ is strongly free of Liouville numbers along b.

For each $n \geqslant n_0$, choose a pseudo-triangulation S_n of X_n such that Γ_{S_n} is quasi-finite. Consider the following assertions:

(1) the sequence $\big(\sum_{x\in\Gamma_{S_n}} \chi(x, \Gamma_{S_n}, \mathscr{F}_{|X_n})\big)_{n\geqslant n_0}$ is eventually constant;

(1') the sequence $\big(\chi_c(X_n) \cdot \mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_{X_n}(\mathscr{F}_{|X_n})\big)_{n\geqslant n_0}$ is eventually constant;

(2) for all $i \geqslant 0$, $\mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F})$ is finite-dimensional.

Then (1) is equivalent to (1') and implies (2). If K is not trivially valued, then (2) implies (1). Moreover, when (1) or (1') is satisfied,

(a) we have

$$\chi_{\mathrm{dR}}(X, \mathscr{F}) = \lim_{n\to+\infty} \big(\chi_c(X_n) \cdot \mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_{X_n}(\mathscr{F}_{|X_n})\big) , \tag{5.1}$$

$$= \lim_{n\to+\infty} \Big(\sum_{x\in\Gamma_{S_n}} \chi(x, \Gamma_{S_n}, \mathscr{F}_{|X_n})\Big) ; \tag{5.2}$$

(b) There is an index n_1 such that for all $n \geqslant n_1$, the following properties hold:

(A) for each $i \geqslant 0$, the natural restriction $\mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n})$ is an isomorphism;

(B) we have

$$\chi_{\mathrm{dR}}(X, \mathscr{F}) = \chi_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}) = \chi_c(X_n) \cdot \mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_{X_n}(\mathscr{F}) . \tag{5.3}$$

Proof. By Corollary **??**, we may extend the scalars, hence assume that K is algebraically closed and not trivially valued.

Let $n \geqslant n_0$. By assumption $\mathscr{F}_{|X_n}$ has well-defined irregularity, hence, by Theorem 4.1.1, $\mathscr{F}_{|X_n}$ has finite-dimensional de Rham cohomology and we have

$$\chi_{\mathrm{dR}}(X_n, \mathscr{F}_{|X_n}) = \chi_c(X_n) \cdot \mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_{X_n}(\mathscr{F}_{|X_n}) = \sum_{x\in\Gamma_{S_n}} \chi(x, \Gamma_{S_n}, \mathscr{F}_{|X_n}) . \tag{5.4}$$

The result now follows from Theorem 1.4.9. □

Under some hypotheses about the analytic skeleta of the curves, the convergence conditions appearing in Theorem 5.1.1 can be made more explicit. We recall that the analytic skeleton Γ_Y of a quasi-smooth K-analytic curve Y is the set of points without neighborhoods isomorphic to a virtual open disk. We refer to (2.3) for the definition of $\chi(x, \Gamma)$.

Lemma 5.1.2. *Let Y be a quasi-smooth K-analytic curve. For each point y of $\Gamma_Y - \partial Y$, we have*

$$N_{\Gamma_Y}(y) \;\geqslant\; 2 \text{ and } \chi(y, \Gamma_Y) \;\leqslant\; 0\;. \tag{5.5}$$

Proof. We may assume that K is algebraically closed and that Y is connected. Assume, by contradiction that there exists a point y in $\Gamma_Y - \partial Y$ such that $N_{\Gamma_Y}(y) \leqslant 1$. If $N_{\Gamma_Y}(y) = 0$, then Γ_Y is reduced to a point because it is connected. Therefore, Y is compact without boundary, hence projective by [Duc, Théorème 3.7.2] and of genus 0. We deduce that it is isomorphism to $\mathbb{P}^{1,\mathrm{an}}_K$, which has empty analytic skeleton, and we get a contradiction. If $N_{\Gamma_Y}(y) = 1$, then we get a contradiction by [Duc, Corollaire 5.1.20].

We deduce that, for each $y \in \Gamma_Y - \partial Y$, we have

$$\chi(y, \Gamma_Y) = 2 - 2g(y) - N_{\Gamma_Y}(y) \leqslant 0.$$

□

Corollary 5.1.3. *In the setting of Theorem 5.1.1, assume moreover that the analytic skeleton Γ_X of X is non-empty and that, for each $n \geqslant n_0$, we have $\Gamma_{S_n} = \Gamma_X \cap X_n$.*

Then, condition (1) is equivalent to

(1") the set of points $x \in \Gamma_X$ such that $\chi(x, \Gamma_X, \mathscr{F}) \neq 0$ is finite.

Moreover, when it holds, for n big enough, we have

$$\sum_{x \in \Gamma_{S_n}} \chi(x, \Gamma_{S_n}, \mathscr{F}_{|X_n}) \;=\; \sum_{x \in \Gamma_X} \chi(x, \Gamma_X, \mathscr{F})\;. \tag{5.6}$$

Proof. By Lemma 5.1.2, for each $x \in \Gamma_X - \partial X$, we have $\chi(x, \Gamma_X) \leqslant 0$, hence, by Lemma 2.2.3, $\chi(x, \Gamma_X, \mathscr{F}) \leqslant 0$. By Lemma 2.2.4, for each $x \in \partial X$, we also have $\chi(x, \Gamma_X, \mathscr{F}) \leqslant 0$. □

5.2. Cuttings.

Let X be a quasi-smooth K-analytic curve. In this section, we introduce the notion of *cutting of X*, which we use as a way to decompose the curve into finite curves. This notion will be important to control the locus where the Liouville condition (about the exponents of a differential equation) arises.

Definition 5.2.1. *A* cutting of X *is a locally finite subset T of X formed by points of type 2 or 3 such that each connected component of $X - T$ is a finite curve.*

Remark 5.2.2. *i) Any pseudo-triangulation of X is a cutting of X.*

ii) As for pseudo-triangulations, a cutting of X can be empty (in which case, every connected component of X is a finite curve).

Definition 5.2.3. *Let T be a cutting of X. Denote by $\mathscr{E}_T$ (resp. $\mathscr{E}_T^\circ$) the set of connected components of $X - T$ (resp. that are not pseudo-disks). We define the* skeleton of T *to be the set*

$$\Gamma_T \;:=\; T \cup \bigcup_{E \in \mathscr{E}_T} \Gamma_E \;=\; T \cup \bigcup_{E \in \mathscr{E}_T^\circ} \Gamma_E\;, \tag{5.7}$$

where Γ_E denotes as usual the analytic skeleton of E.

Notice that if S is a pseudo-triangulation (and hence also a cutting) the skeleton of S as a cutting coincides with the skeleton Γ_S as a pseudo-triangulation (i.e. the above definition agrees with [PP13, Definition 1.1.10]).

The following lemmas establish the relation between pseudo-triangulations and cuttings.

Lemma 5.2.4. *Let T be a cutting of X. For each $E \in \mathscr{E}_T^\circ$, let S_E be a pseudo-triangulation of E that is a locally finite subset of X and such that $\Gamma_{S_E} = \Gamma_E$. Then, the set $S_T := T \cup \bigcup_{E\in\mathscr{E}_T^\circ} S_E$ is a pseudo-triangulation of X whose skeleton is Γ_T.*

In particular, if Γ_T meets every connected component of X, then there exists a continuous proper retraction by deformation $r_T : X \to \Gamma_T$.

Proof. We may assume that X is connected. Let $x \in T$. Denote by $\mathscr{E}_x^\circ$ the subset of $\mathscr{E}_T^\circ$ made of the elements whose closure meets x. Since almost every connected component of $X - \{x\}$ is a virtual open disk, the set $\mathscr{E}_x^\circ$ is finite. It follows that S_T is finite in the neighborhood of x.

Noting that S_T is obviously finite around points of $X - T$, the previous argument shows that S_T is locally finite. The other properties of a pseudo-triangulation are clear, as well as the statement about the skeleton. □

Lemma 5.2.5. *Let T be a cutting of X and let $T' := T \cap \Gamma_X$. Then*

i) T' is again a cutting of X and it satisfies $\Gamma_{T'} = \Gamma_X$;

ii) Assume that Γ_X is not empty (or equivalently that X is not a pseudo-disk nor the projective line). If T is a pseudo-triangulation of X, then T' is a pseudo-triangulation too.

In particular, by Lemma 5.2.4, if T is a cutting and if Γ_X is not empty, there exists a pseudo-triangulation S containing T' such that $\Gamma_S = \Gamma_{T'} = \Gamma_X$.

Proof. We can assume that X is connected. If Γ_X is empty, then X is the projective line or a pseudo disk. In particular it is a finite curve and i) holds. We now assume that Γ_X is not empty.

In order to prove the first part of the claim, it suffices to show that every element of $\mathscr{E}_{T'}$ is a finite curve. We actually have a natural surjective map $\varphi : \mathscr{E}_T \to \mathscr{E}_{T'}$ associating to a connected component E of $X - T$ the connected component of $X - T'$ containing E. Let $E' \in \mathscr{E}_{T'}$.

If the set $\varphi^{-1}(E')$ is reduced to an individual element, then E' is a connected component of $X - T$ and it is therefore a finite curve.

Assume that $\varphi^{-1}(E')$ is not reduced to an individual element. By construction, E' is disjoint union of $T \cap E'$ with the connected components E of $X - T$ that meet E':

$$E' \;=\; (T \cap E') \bigcup \Big(\bigcup_{E\subseteq E'} E \Big)\,. \tag{5.8}$$

We distinguish two cases:

(a) $\varphi^{-1}(E')$ contains a connected component E of $X - T$ that meets Γ_X.

(b) $\varphi^{-1}(E')$ do not contains a connected component as above.

We firstly consider case (a). Denote by $\overline{E}$ the closure of E in X and let $x \in \overline{E} - E$. Necessarily, $x \in T$ and it is then a point of type (2) or (3). Moreover, if $x \notin \Gamma_X$, then every virtual open disk D in X having x in the closure is contained in E'. We deduce that, if $x \notin \Gamma_X$, then the whole connected component of $X - E$ having x in the boundary (necessarily a virtual closed disk) is contained in E'.

On the other hand, if $x \in \Gamma_X$, then $x \in T'$ and hence necessarily lies in $\overline{E'} - E'$. We deduce that E is the unique element of $\mathscr{E}^\circ$ contained in E' and that E' is obtained from E adding to it the points of T that lie in $\mathscr{T}_E := (\overline{E} - E) - \Gamma_X$ together with all virtual open disks D in X whose closure meets $\mathscr{T}_E$. Notice that $\mathscr{T}_E$ is a finite set because E is finite and the points of $\mathscr{T}_E$ lie at the closure of the open boundary of E.

Now, E being a finite curve, we can consider a finite pseudo-triangulation S_E of E. We set $S_{E'} := S_E \cup \mathscr{T}_E$. Clearly, $(\Gamma_X \cap E) \subseteq \Gamma_E$ and for every $x \in \mathscr{T}_E$ the segment joining x to Γ_X in contained in Γ_E. It follows that $E' - S_{E'}$ is finite disjoint union of virtual open disks or open pseudo-annuli and that $S_{E'}$ is a pseudo-triangulation of E'. Moreover, $S_{E'}$ is finite because so are S_E and $\mathscr{T}_E$. It follows that E' is a finite curve.

Consider now case (b). Since E' does not meet Γ_X, necessarily it is contained in a virtual open disk D whose boundary meet Γ_X. Since $T' \cap D = \emptyset$ we must have $E' = D$ which proves that E' is a finite curve.

We now prove ii). In this case, reasoning as above, if $\varphi^{-1}(E')$ counts more than one element, then E' is a virtual open disk whose closure meets Γ_X. Otherwise, if $\varphi^{-1}(E)$ consists in one element, then $E' = E$ is an open pseudo-annulus. The claim follows. □

5.3. A density result.

In order to apply Theorem 5.1.1 and Corollary 5.1.3, we will have to ensure that a certain density condition is satisfied. To do so, we will use a result reminiscent of Behnke-Stein's theorem in the complex analytic setting (see [BS49]).

Lemma 5.3.1. *Let Y be a connected K-analytic space. For any $r \in \mathbb{R}_+^* - \sqrt{|K^*|}$, the space $Y_r := Y \hat{\otimes}_K K_r$ (see (??)) is connected.*

Proof. Consider the projection map $\pi : Y_r \to Y$. For each point y in Y, the fiber $\pi^{-1}(y) \simeq \mathcal{M}(\mathscr{H}(y) \hat{\otimes}_K K_r)$ is an annulus over $\mathscr{H}(y)$. In particular, it is connected. Let us denote by y_r its Shilov boundary.

The result follows from the fact that the map $y \in Y \mapsto y_r \in Y_r$ is a continuous section of π (see [Ber90, Corollary 5.2.7]).

□

The basic result we use is a direct consequence of one by Liu and van der Put (see [LvdP95, Lemma 3.5]).

As usual, if Y is a K-analytic space, we denote by ∂Y its boundary in the sense of Berkovich (see [Ber90, Definition 2.5.7] and [Ber93, Definition 1.5.4]). When Y is a K-affinoid curve, ∂Y coincides with the Shilov boundary of Y (see [Meh18, Lemma 2.4]). In particular, it is a finite set. Moreover, if Y is an analytic domain of a K-analytic space X, then, by [Ber93, Proposition 1.5.5], the topological boundary of Y in X coincides with $\partial(Y/X)$, which is a subset of ∂Y.

Remark 5.3.2. *The following result corresponds to [LvdP95, Lemma 3.8] where the assumption is phrased in terms of the reductions of U and V. Since working out the translation requires some work, it seemed better to write down the proof.*

Lemma 5.3.3. *Let V be a quasi-smooth K-affinoid space and let U be an affinoid domain of V such that each connected component of $V - U$ contains a point of the boundary ∂V of V. Then, the restriction map $\mathscr{O}(V) \to \mathscr{O}(U)$ has dense image.*

Proof. To prove the result, we claim that we may assume that K is not trivially valued and that U and V are strictly K-affinoid. Indeed, let $r \in \mathbb{R}_+^* - \sqrt{|K^*|}$. By Lemma **??**, it is enough to prove the result for the map $\mathscr{O}(V) \hat{\otimes}_K K_r \to \mathscr{O}(U) \hat{\otimes}_K K_r$, that is to say for the map $\mathscr{O}(V_r) \to \mathscr{O}(U_r)$ (cf. Corollary **??**).

Let E be a connected component of $V - U$. By Lemma 5.3.1, E_r is connected, hence it is a connected component of $V_r - U_r$ and the connected components of $V_r - U_r$ are of this form. By

[Ber90, Corollary 2.5.12], E_r contains a point of ∂V_r. It follows that we may replace K by K_r in order to prove the result. In particular, we may assume that K is non-trivially valued. By repeating this procedure finitely many times, we may also assume that U and V are strictly K-affinoid. We may also assume that V is connected.

By [vdP80, Theorem 1.1], V embeds as an affinoid domain of the analytification C^{an} of a connected smooth projective curve C over K. Let D be a connected component of $C^{\mathrm{an}} - U$. By assumption, it contains a point of boundary ∂V of V. It now follows from the description of the neighborhoods of type 2 points that D contains a rigid point y_D that does not belong to V.

By [LvdP95, Lemma 3.5], $C^{\mathrm{an}} - U$ has finitely many connected components $D_1, \dots, D_m$ and the restriction map $\mathscr{O}(C^{\mathrm{an}} - \{p_{D_1}, \dots, p_{D_m}\}) \to \mathscr{O}(U)$ has dense image. Since, by construction, V is contained in $C^{\mathrm{an}} - \{p_{D_1}, \dots, p_{D_m}\}$, the result follows. □

In the following, we will use the strong notion of triangulation of A. Ducros (see [Duc, 5.1.13], [PP13, Definition 1.1.4]). In particular, each connected component of $X - S$ is relatively compact in X, hence each edge of the skeleton Γ_S (i.e. each connected component of $\Gamma_S - S$) is relatively compact in X.

Theorem 5.3.4. *Let Y be a connected quasi-smooth K-analytic curve that is not proper and let V be an affinoid domain of Y.*

Let $\mathscr{E}$ be the set of connected components of $Y - V$ that are relatively compact in Y. Assume that every element of $\mathscr{E}$ meets the boundary ∂Y of Y.

Then, the restriction map $\mathscr{O}(Y) \to \mathscr{O}(V)$ has dense image.

Proof. Since V is closed, the topological boundary of each connected component of $Y - V$ in Y is contained in V, hence in the boundary of V in Y. It follows that each element of $\mathscr{E}$ contains at least a germ of segment out of a point of ∂V that is not contained in V. Since ∂V is a finite set and, for each $x \in \partial V$, the number of germ of segments out of x that are not in V is finite too, $\mathscr{E}$ is a finite set. We set $V_1 := V \cup \bigcup_{E\in\mathscr{E}} E$. It is a compact analytic domain of Y, hence an affinoid domain by [Duc, Théorème 6.1.3]. By assumption, each element E of $\mathscr{E}$ contains a point of ∂Y, which is necessarily a point of ∂V_1, hence, by Lemma 5.3.3, the restriction map $\mathscr{O}(V_1) \to \mathscr{O}(V)$ has dense image.

By [Duc, Théorème 5.1.14], there exists a triangulation S' of V_1 and a triangulation S of Y containing S'. The set $S \cap V_1$ is a triangulation of V_1 satisfying $\Gamma_{S\cap V_1} = \Gamma_S \cap V_1$. It is an easy consequence of the fact that each connected component of $Y - \partial V_1$ (hence of $Y - S$) is either contained in V_1 or in its complement.

In particular, we have $r_S^{-1}(\Gamma_S \cap V_1) \supseteq V_1$, where $r_S \colon Y \to \Gamma_S$ denotes the retraction associated to S. Actually, we have $r_S^{-1}(\Gamma_S \cap V_1) = V_1$. Indeed, if it were not the case, there would exist a virtual open disk D not contained in V_1 but with boundary point z in V_1, hence a connected component of $Y - V_1$ that is relatively compact in Y, which does not exist.

Since each edge of the skeleton Γ_S is the skeleton of a virtual open annulus, it is canonically endowed with a length. It then follows from [Duc, Proposition 1.6.3] that Γ_S is paracompact and, in particular, that S is countable.

We claim that there exists an increasing sequence $(\Gamma_n)_{n\geqslant 1}$ of finite subgraphs of Γ_S such that

i) $\bigcup_{n\geqslant 1} \Gamma_n = \Gamma_S$;
ii) $\Gamma_1 = \Gamma_S \cap V_1$;

and, for all $n \geqslant 1$,

iii) the topological relative boundary of Γ_n in Γ_S is contained in S;

iv) $\Gamma_S - \Gamma_n$ has no connected component that is relatively compact in Γ_S.

We construct the sequence $(\Gamma_n)_{n\geqslant 1}$ by induction. We set $\Gamma_1 := \Gamma_S \cap V_1$. For each $n \geqslant 1$, we define Γ'_{n+1} to be the union of Γ_n and all the (finitely many) edges of Γ_S meeting Γ_n and we define Γ_{n+1} to be the union of Γ'_{n+1} and all the relatively compact connected components of $\Gamma_S - \Gamma'_{n+1}$. It is easy to check that all the required properties hold. For instance, i) follows from the countability of S and the connectedness of Γ_S.

For each $n \geqslant 1$, set $V_n := r_S^{-1}(\Gamma_n)$. Let $n \geqslant 1$ and let E be a connected component of $V_{n+1} - V_n$. Its inverse image Γ_E by r_S is a connected component of $\Gamma_{n+1} - \Gamma_n$. It is contained in a connected component of $\Gamma_S - \Gamma_n$ but cannot coincide with it, by property iv), since it is relatively compact in Γ_S. It follows that $E = r_S^{-1}(\Gamma_E)$ contains a point of the topological boundary of V_{n+1} in Y, hence a point of ∂V_{n+1}. By Lemma 5.3.3, the restriction map $\mathscr{O}(V_{n+1}) \to \mathscr{O}(V_n)$ has dense image.

By property i), Y is exhausted by the V_n's, hence $\mathscr{O}(Y) \xrightarrow{\sim} \varprojlim \mathscr{O}(V_n)$ and the result follows.

Let us be more precise. Let $f_0 \in \mathscr{O}(V)$ and let $\varepsilon > 0$. Set $V_0 = V$. Using the density of $\mathscr{O}(V_{n+1})$ in $\mathscr{O}(V_n)$ for each $n \geqslant 0$, we construct by induction a family $(f_n)_{n\geqslant 0} \in \prod_{n\geqslant 0} \mathscr{O}(V_n)$ such that, for each $n \geqslant 0$, $\|f_{n+1} - f_n\|_{V_n} \leqslant \varepsilon/(n+1)$.

For each $n \geqslant 0$, the sequence $((f_m)_{|V_n})_{m\geqslant n}$ is a Cauchy sequence in $\mathscr{O}(V_n)$, hence it converges to an element $g_n \in \mathscr{O}(V_n)$. It is clear that, for each $n, n' \geqslant 0$ with $n' \geqslant n$, we have $(g_{n'})_{|V_n} = g_n$. It follows that the g_n's may be glued into a global function $g \in \varprojlim \mathscr{O}(V_n) = \mathscr{O}(Y)$. Moreover, we have $\|g_0 - f\|_V \leqslant \max_{n\geqslant 0} \|f_{n+1} - f_n\|_V \leqslant \varepsilon$. The result follows. □

Corollary 5.3.5. *Let Y be a connected quasi-smooth K-analytic curve that is not proper and let U be an analytic domain of Y.*

Let $\mathscr{E}$ be the set of connected components of $Y - U$ that are relatively compact in Y. Assume that each element of $\mathscr{E}$ meets the boundary ∂Y of Y.

Then, the restriction map $\mathscr{O}(Y) \to \mathscr{O}(U)$ has dense image.

Proof. By definition of compact convergence, it is enough to prove that, for each $\varepsilon > 0$, each $f \in \mathscr{O}(U)$ and each compact subset X of U, there exists $g \in \mathscr{O}(Y)$ such that $\|f - g\|_X < \varepsilon$. For this, it is enough to prove that each compact subset X of U is contained in an affinoid domain V of U such that the restriction $\mathscr{O}(Y) \to \mathscr{O}(V)$ has dense image.

Let X be a compact subset of U. We may cover X by choosing, for every point x of X, an affinoid neighborhood of x in U. By compactness, X is actually covered by finitely many of them, hence it is contained in a compact analytic domain V' of U. Since Y is not proper, V' is not proper either, hence it is affinoid, by [Duc, Théorème 6.1.3].

Let $\mathscr{E}_U$ be the set of connected components of $U - V'$ that are relatively compact in U. By the same argument as in the proof of Theorem 5.3.4, it is a finite set. It follows that $V := V' \cup \bigcup_{E\in\mathscr{E}_U} E$ is a compact analytic domain of U, hence an affinoid domain of U, by [Duc, Théorème 6.1.3] again. Moreover, by construction, no connected component of $U - V$ is relatively compact in U.

We now prove that the assumptions of Theorem 5.3.4 are satisfied by V and therefore that the map $\mathscr{O}(Y) \to \mathscr{O}(V)$ has dense image. Let C be a connected component of $Y - V$ that is relatively compact in Y and that contains no points of ∂Y. Let $\overline{C}$ (resp. B) be the topological closure (resp. topological boundary) of C in Y. By assumption $\overline{C}$ is a compact subset of Y. The set B is contained in V, hence in U. It follows that $C \cap (Y - U) = \overline{C} \cap (Y - U)$ and, in particular, that $C \cap (Y - U)$ is closed in $Y - U$. Since C is open in $Y - V$, $C \cap (Y - U)$ is also open in $Y - U$. It follows that $C \cap (Y - U)$ is a disjoint union of connected components of $Y - U$. Let Q be one of those connected components. The closure of Q in Y is closed in the compact $\overline{C}$, hence it is itself compact. Moreover, by assumption, Q contains no points of ∂Y. It follows that such a connected component cannot exist, hence $C \cap (Y - U)$ is empty and C is contained in U. In particular, C is a connected component of $U - V$ that is relatively compact in U. By definition of V, there are no such connected components.

We have just proved that there are no connected components of $Y - V$ that are relatively compact in Y and contain no points of ∂V. By Theorem 5.3.4, the restriction map $\mathscr{O}(Y) \to \mathscr{O}(V)$ has dense image. Since X is contained in V, the result follows. □

5.4. Cuttings and de Rham cohomology.

We finally present some conditions on a cutting of a curve X ensuring that it admits an exhaustion as in Theorem 5.1.1 and Corollary 5.1.3. Recall that we introduced some terminology for graphs in [PP13, Section 1.1.7].

We first introduce two constructions.

Notation 5.4.1. *Let Γ be a graph and T be a subset of Γ. Let Δ be a subgraph of Γ.*

We denote by $\mathscr{E}_T(\Delta)$ the set of connected components of $\Gamma - T$ whose closure meets Δ and set

$$\Delta_T^o := \Delta \cup \bigcup_{E \in \mathscr{E}_T(\Delta)} E.$$

We denote by $\mathscr{E}^c(\Delta)$ the set of connected components of $\Gamma - \Delta$ that are relatively compact in Γ and set

$$\Delta^c := \Delta \cup \bigcup_{E \in \mathscr{E}^c(\Delta)} E.$$

Lemma 5.4.2. *Let Γ be a connected locally finite graph and let T be a locally finite subset of Γ. Let Δ be a quasi-finite subgraph of Γ.*

Then, the following properties hold:

i) The sets $\mathscr{E}_T(\Delta)$ and $\mathscr{E}^c(\Delta)$ are finite and their elements are finite unions of points of T and connected components of $\Gamma - T$;

ii) The set Δ_T^o and Δ^c are connected.

iii) The set Δ_T^o is open.

iv) The set Δ^c is a neighborhood of $\Delta^c \cap (\overline{\Delta} - \Delta)$. In particular, if Δ is open, then Δ^c is open.

Proposition 5.4.3. *Let Γ be a paracompact connected locally finite graph.*

Let T be a locally finite subset of Γ such that each connected component of $\Gamma - T$ is a quasi-finite graph. Then, there exists a non-decreasing sequence $(\Gamma_n)_{n\geqslant 0}$ of subgraphs of Γ covering Γ such that, for each $n \geqslant 0$,

i) Γ_n is open in Γ and connected;

ii) Γ_n is a finite union of points of T and connected components of $\Gamma - T$;

iii) no connected component of $\Gamma - \Gamma_n$ is relatively compact in Γ;

iv) no connected component of $\Gamma_{n+1} - \Gamma_n$ is relatively compact in Γ_{n+1}.

Proof. If Γ is quasi-finite, then the result holds with the constant sequence equal to Γ. From now on, we assume that Γ is not quasi-finite. Note that the fact that it is locally finite then forces T to be infinite.

By induction, we construct a sequence of subgraphs $(\Gamma_n)_{n\geqslant 0}$ of Γ such that, for each $n \geqslant 0$, Γ_n satisfies properties i), ii), iii) of the statement as well as iv') no connected component of $\Gamma_n - \Gamma_{n-1}$ is relatively compact in Γ_n

when $n \geqslant 1$.

To construct Γ_0, we pick any point $t \in T$ and set $\Gamma_0 := (\{t\}^o)^c$. By construction, it satisfies property iii) and, by Lemma 5.4.2, it also satisfies properties i) and ii).

Let $n \geqslant 0$ and assume that we have already constructed Γ_n with the required properties. Note that its closure $\overline{\Gamma}_n$ in Γ is then a finite union of points of T and connected components of $\Gamma - T$, hence a quasi-finite graph. We set $\Gamma_{n+1} := (\overline{\Gamma}_n^o)^c$. As before, it satisfies property iii) by construction, and properties i) and ii) by Lemma 5.4.2.

Let us finally prove property iv') in the case where $n \geqslant 0$. Assume, by contradiction, that there exists a connected component C of $\Gamma_{n+1} - \Gamma_n$ that is relatively compact in Γ_{n+1}. Since Γ_{n+1} is open in Γ, C is open in $\Gamma - \Gamma_n$. Since Γ_n is open in Γ_{n+1}, C is compact, hence closed in $\Gamma - \Gamma_n$, hence a connected component of $\Gamma - \Gamma_n$, which is absurd.

We have proven all the wanted properties except for the fact that $(\Gamma_n)_{n\geqslant 0}$ covers Γ. It is a consequence the countability of T, which follows from paracompactess.

□

Notation 5.4.4. *For each subset L of X, we set*

$$\mathscr{B}_{\to}(L) := \bigcup_U \partial^o U,$$

where U runs through the connected open subsets of X containing L.

Example 5.4.5. *Let x be a point in $\mathbb{P}^{1,\mathrm{an}}_K$. For each $y \in \mathbb{P}^{1,\mathrm{an}}_K$, there are infinitely many germs of segments emanating from y, but only one in $\mathscr{B}_{\to}(x)$.*

Remark 5.4.6. *Let L be a subset of X and let $y \in X - L$. If L meets at least two connected components of $X - \{y\}$, then $\mathscr{B}_{\to}(L)$ contains no germs of segment emanating from y.*

Let C be a connected component of $X - \{y\}$. If C does not meet L, then $\partial^o C \cap \mathscr{B}_{\to}(L) = \emptyset$. If C contains L, then the set of germs of segments emanating from y that belong to $\mathscr{B}_{\to}(L)$ is finite.

Recall, in the case of finite curves (in particular for proper curves and curves with empty analytic skeleton), we already characterized the finiteness of the de Rham cohomology and computed the index in Theorem 4.1.1.

Theorem 5.4.7. *Assume that X is connected, not proper and that its analytic skeleton Γ_X is non-empty. Assume that, for each $x \in \partial X$, the radii of convergence of $\mathscr{F}$ are spectral non-solvable at x and that, for each $b \in \mathrm{Sing}_m(x, \mathscr{F})$, $\mathscr{F}$ is strongly free of Liouville numbers along b. Let L be a compact subset of X. Let T be a cutting of X such that*

i) for each connected component C of $X - T$ and each $b \in \partial^o C \cap \mathscr{B}_{\to}(L)$, the total height of the Newton polygon of $\mathscr{F}$ is log-linear along b and $\mathscr{F}$ is strongly free of Liouville numbers along b;

ii) $\Gamma_T = \Gamma_X$ (or equivalently $T \subseteq \Gamma_X$).

Then the following assertions are equivalent:

(1) the set of points $x \in \Gamma_X$ such that $\chi(x, \Gamma_X, \mathscr{F}) \neq 0$ is finite;

(2) for all $i \geqslant 0$, $\mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F})$ is finite-dimensional.

(a) we have

$$\chi_{\mathrm{dR}}(X, \mathscr{F}) = \sum_{x \in \Gamma_X} \chi(x, \Gamma_X, \mathscr{F}). \tag{5.9}$$

(b) there exists a connected open subset X' of X such that

(A) X' is a finite curve with $\Gamma_{X'} \subseteq \Gamma_X$;

(B) the natural restriction $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X',\mathscr{F}_{|X'})$ is an isomorphism for each $i \geqslant 0$;

(C) we have

$$\chi_{\mathrm{dR}}(X,\mathscr{F}) \;=\; \chi_{\mathrm{dR}}(X',\mathscr{F}_{|X'}) \;=\; \chi_c(X')\cdot \mathrm{rank}(\mathscr{F}_{|X'}) - \mathrm{Irr}_{X'}(\mathscr{F})\;. \tag{5.10}$$

Proof. By assumption Γ_X is not empty, hence we have a well-defined retraction $r : X \to \Gamma_X$. Recall that X is paracompact by [Duc, Théorème 4.5.10], hence Γ_X is paracompact too. We may then apply Propostion 5.4.3 to Γ_X and T to find a sequence $(\Gamma_n)_{n\geqslant 0}$ of subgraphs of Γ_X that satisfy the conditions i) to iv) of that proposition.

For each $n \geqslant 0$, set $X_n := r^{-1}(\Gamma_n)$. It is a connected open subset of X with analytic skeleton Γ_n. Since X_n is a finite union of elements of T and of connected components of $X - T$, X_n is a finite curve.

Since L is compact, there exists $n_0 \geqslant 0$ such that $r^{-1}(L) \subseteq \Gamma_{n_0}$, hence $L \subseteq X_{n_0}$. Let $n \geqslant n_0$. Then each element b of $\partial^o X_n$ belongs to $\mathscr{B}_{\to}(L)$ and to the open boundary of some connected component of $X - T$ (because X_n is a finite union of elements of T and of connected components of $X - T$). In particular, for such a b, the total height of the Newton polygon of $\mathscr{F}$ is log-linear along b and $\mathscr{F}$ is strongly free of Liouville numbers along b. Moreover, since Γ_n is open in Γ, X_n is open in X too, hence $\partial X_n \subseteq \partial X$ by [Ber93, Proposition 1.5.5]. In particular, for each $x \in \partial X_n$, the radii of convergence of $\mathscr{F}_{|X_n}$ are spectral non-solvable at x and that, for each $b \in \mathrm{Sing}_m(x,\mathscr{F}_{|X_n})$, $\mathscr{F}_{|X_n}$ is strongly free of Liouville numbers along b.

Finally, by construction, $\Gamma_{n+1} - \Gamma_n$ has no connected component that is relatively compact in Γ_{n+1}, hence $X_{n+1} - X_n$ has no connected component that is relatively compact in X_{n+1}, and, by Corollary 5.3.5, the restriction map $\mathscr{O}(X_{n+1}) \to \mathscr{O}(X_n)$ has dense image.

If K is non-trivially valued, then the result follows from Corollary 5.1.3.

If K is trivially valued, then, by [**?**, Lemma 1.1.63], X contains at most one point of type 2, hence (1) holds, and so does (2), by Corollary 5.1.3.

□

Remark 5.4.8. *It follows from Remark 5.4.6 that the set of germs of segments b along which $\mathscr{F}$ is required to be free of Liouville numbers is locally finite.*

We add a final result concerning the trivial differential equation.

Corollary 5.4.9. *Assume that X is connected and boundaryless. Then $\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{O})$ is finite-dimensional if, and only if, X is a finite curve.*

Moreover, in this case, we have

$$\chi_{\mathrm{dR}}(X,\mathscr{O}) \;=\; \chi_c(X)\;. \tag{5.11}$$

Proof. Let us first assume that X is a finite curve. Then the result follow from Corollary 4.1.2.

Let us now assume that $\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{O})$ is finite-dimensional. If X is proper or if its analytic skeleton is empty, then X is finite, so we may assume that this is not the case. Similarly, if K is trivially valued, then X is finite by [**?**, Lemma 1.1.63], so we may assume that this is not the case. By REF, we may also assume that K is algebraically closed.

Any cutting T of X such that $\Gamma_T = \Gamma_X$ satisfies the assumptions of Theorem 5.4.7. We deduce that the set of points $x \in \Gamma_X$ such that $\chi(x,\Gamma_X,\mathscr{O}) \neq 0$ is finite. For each $x \in \Gamma_X$, we have

$$\chi(x,\Gamma_X,\mathscr{O}) = \chi(x,\Gamma_X) = 2 - 2g(x) - N_{\Gamma_X}(x). \tag{5.12}$$

Recall that, by Lemma 5.1.2, we have $N_{\Gamma_X}(x) \geqslant 2$. It follows that Γ_X contains only finitely many points x such that $g(x) > 0$ or $N_{\Gamma_X}(x) \geqslant 3$ (that is to say bifurcation points of Γ_x). In particular, the graph Γ_X is quasi-finite. The result now follows from [**?**, Lemma 1.1.61]. □

References

Bal87 F. Baldassarri, *Comparaison entre la cohomologie algébrique et la cohomologie p-adique rigide à coefficients dans un module différentiel. I. Cas des courbes*, Invent. Math. **87** (1987), no. 1, 83–99. MR 862713 (88d:14014)

Ber90 Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.

Ber93 ———, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. (1993), no. 78, 5–161 (1994). MR MR1259429 (95c:14017)

Bou12 N. Bourbaki, *Algèbre. Chapitre 8: Modules et anneaux semi simples*, Springer, Berlin, 2012.

BP18 Velibor Bojković and Jérôme Poineau, *Pushforwards of p-adic differential equations*, Amer. J. Math. (2018), to appear.

BS49 Heinrich Behnke and Karl Stein, *Entwicklung analytischer Funktionen auf Riemannschen Flächen*, Math. Ann. **120** (1949), 430–461. MR 0029997 (10,696c)

CM97 G. Christol and Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques. II*, Ann. of Math. (2) **146** (1997), no. 2, 345–410. MR 1477761 (99a:12009)

CM00 ———, *Sur le théorème de l'indice des équations différentielles p-adiques. III*, Ann. of Math. (2) **151** (2000), no. 2, 385–457. MR 1765703 (2001k:12014)

CM01 ———, *Sur le théorème de l'indice des équations différentielles p-adiques. IV*, Invent. Math. **143** (2001), no. 3, 629–672. MR 1817646 (2002d:12005)

Del70 Pierre Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970. MR MR0417174 (54 #5232)

Duc Antoine Ducros, *La structure des courbes analytiques*, http://www.math.jussieu.fr/~ducros/livre.html.

Gro54 A. Grothendieck, *Espaces vectoriels topologiques*, Instituto de Matemática Pura e Aplicada, Universidade de São Paulo, São Paulo, 1954. MR 0077884 (17,1110a)

Gro61 ———, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167. MR 0217085 (36 #177c)

Gro67 ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 0238860 (39 #220)

HTT08 Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi. MR 2357361 (2008k:32022)

Kat87 Nicholas M. Katz, *On the calculation of some differential Galois groups*, Invent. Math. **87** (1987), no. 1, 13–61. MR MR862711 (88c:12010)

Ked10 Kiran S. Kedlaya, *p-adic differential equations*, Cambridge Studies in Advanced Mathematics, vol. 125, Cambridge Univ. Press, 2010.

Ked15a Kiran S; Kedlaya, *Convergence polygons for connections on non archimedean curves*, arXiv, 2015, http://arxiv.org/pdf/1505.01890v2.pdf, pp. 1–40.

Ked15b Kiran S. Kedlaya, *Local and global structure of connections on nonarchimedean curves*, Compos. Math. **151** (2015), no. 6, 1096–1156. MR 3357180

Köp74 Ursula Köpf, *Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen*, Schr. Math. Inst. Univ. Münster (2) (1974), no. Heft 7, iv+72. MR MR0422671 (54 #10657)

LvdP95 Q. Liu and M. van der Put, *On one-dimensional separated rigid spaces*, Indag. Math. (N.S.) **6** (1995), no. 4, 439–451. MR 1365187

Mal74 Bernard Malgrange, *Sur les points singuliers des équations différentielles*, Enseignement Math. (2) **20** (1974), 147–176. MR 0368074 (51 #4316)

Mat02 Shigeki Matsuda, *Katz correspondence for quasi-unipotent overconvergent isocrystals*, Compositio Math. **134** (2002), no. 1, 1–34. MR MR1931960 (2003j:12007)

Meh18 Vlerë Mehmeti, *Patching over Berkovich Curves and Quadratic Forms*, arXiv, 2018, https://arxiv.org/abs/1711.00341.

Poi10 Jérôme Poineau, *Raccord sur les espaces de Berkovich*, Algebra Number Theory **4** (2010), no. 3, 297–334. MR 2602668

PP13 Andrea Pulita and JÃ©rÃ´me Poineau, *The convergence Newton polygon of a p-adic differential equation III : global decomposition and controlling graphs*, arxiv, 2013, http://arxiv.org/abs/1308.0859, pp. 1–81.

PP15 Jérôme Poineau and Andrea Pulita, *The convergence Newton polygon of a p-adic differential equation. II: Continuity and finiteness on Berkovich curves.*, Acta Math. **214** (2015), no. 2, 357–393 (English).

Pul07 Andrea Pulita, *Rank one solvable p-adic differential equations and finite abelian characters via Lubin-Tate groups*, Math. Ann. **337** (2007), no. 3, 489–555. MR MR2274542

Pul13 Andrea Pulita, *An algorithm computing non-solvable spectral radii of p-adic differential equations.*, C. R., Math., Acad. Sci. Paris **351** (2013), no. 5-6, 167–171 (English).

Pul15 ______, *The convergence Newton polygon of a p-adic differential equation. I: Affinoid domains of the Berkovich affine line.*, Acta Math. **214** (2015), no. 2, 307–355 (English).

Sch02 Peter Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. MR 1869547 (2003a:46106)

vdP80 Marius van der Put, *The class group of a one-dimensional affinoid space*, Ann. Inst. Fourier (Grenoble) **30** (1980), no. 4, 155–164. MR 599628

You92 Paul Thomas Young, *Radii of convergence and index for p-adic differential operators*, Trans. Amer. Math. Soc. **333** (1992), no. 2, 769–785. MR 1066451 (92m:12015)

Jérôme Poineau jerome.poineau@unicaen.fr
Laboratoire de MathÃ©matiques Nicolas Oresme, Université de Caen Normandie, BP 5186, F - 14032 Caen Cedex

Andrea Pulita andrea.pulita@univ-grenoble-alpes.fr
Université Grenoble Alpes, Institut Fourier, CS 40700, 38058 Grenoble cedex 9