Non-archimedean compactifications of complex analytic spaces

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(joint work with Marco Maculan)

Let X be a complex algebraic variety, *i.e.* a separated scheme of finite type over $\mathbf{C}$, and let X^{h} be its analytification. We would like to construct a compactification of X^{h} that is canonical in some sense. This is not possible to achieve in the category of complex analytic spaces, so our aim will be to find a compact locally ringed space $X^{\urcorner}$ with an open embedding $X^{\mathrm{h}} \hookrightarrow X^{\urcorner}$.

1. A valuative boundary

We denote by $\mathbf{C}_0$ the field $\mathbf{C}$ endowed with the trivial absolute value $|\cdot|_0$. We will work in the category of analytic spaces over $\mathbf{C}_0$ in the sense of V. Berkovich (see [2, 3]). Recall that we have an analytification functor $X \mapsto X_0^{\mathrm{an}}$ from algebraic varieties over $\mathbf{C}$ to analytic spaces over $\mathbf{C}_0$.

In the affine case $X = \mathrm{Spec}(A)$, where A is an algebra of finite type over $\mathbf{C}$, X_0^{an} may be defined as the set of multiplicative seminorms on A that induce the trivial absolute value $|\cdot|_0$ on $\mathbf{C}$ endowed with the weak topology. It is also endowed with a sheaf of analytic functions. The general case may be obtained from the affine case by glueing.

Starting with an algebraic variety X over $\mathbf{C}$, there is another natural way to associate an analytic space over $\mathbf{C}_0$. Endowing $\mathbf{C}$ with the discrete topology, one may consider X as a formal scheme and consider its generic fiber in the sense of Raynaud. Following [7], we will denote it by $X^{\beth}$. It is a compact subset of X_0^{an}. In the affine case $X = \mathrm{Spec}(A)$, we have

$$X^{\beth} = \{x \in X_0^{\mathrm{an}} : |f(x)| \leq 1, f \in A\}.$$

We may now define the non-archimedean boundary of X by

$$X_\infty := X_0^{\mathrm{an}} - X^{\beth}.$$

It may be identified with the set of seminorms that have no center on X.

It is interesting to remark that, if X is embedded as an open subset in a proper algebraic variety Y over $\mathbf{C}$ with complement Z, then X_∞ may be identified with the generic fiber (in the sense of Raynaud–Berthelot) of the formal completion $\hat{Y}_Z$ of Y along Z deprived of (the analytication of) its special fiber Z. In particular, the latter construction does not depend on the choice of Y.

This set was first defined by Berkovich in a letter to V. Drinfeld and subsequentely used by O. Ben–Bassat and M. Temkin in [1] to prove some descent results (reconstructing coherent sheaves on Y from coherent sheaves on $\hat{Y}_Z$ and X). It was also independently defined by A. Thuillier in [7], where he proved that if Y is regular and Z has normal crossings, then the dual complex of Z is homotopy equivalent to X_∞. As a consequence, the homotopy type of the dual complex of the boundary depends only on X and not on the chosen compactification.

2. Hybrid spaces

In order to put together the spaces X^h and X_∞, we need a "hybrid" space that contains both usual complex analytic spaces and analytic spaces over $\mathbf{C}_0$.

Denote by $\mathbf{C}_{\mathrm{hyb}}$ the field $\mathbf{C}$ endowed with the norm $\|\cdot\|_{\mathrm{hyb}} := \max(|\cdot|_0, |\cdot|_\infty)$, where $|\cdot|_\infty$ is the usual absolute value on $\mathbf{C}$. It is a Banach ring. As a consequence, the theory developed in [2] provides us with a definition of analytic space over $\mathbf{C}_{\mathrm{hyb}}$ and an analyfication functor $X \mapsto X^{\mathrm{hyb}}$.

In the affine case $X = \mathrm{Spec}(A)$, the definition is close to the usual one: X^{hyb} may be defined as the set of multiplicative seminorms on A that are bounded by the norm $\|\cdot\|_{\mathrm{hyb}}$ on $\mathbf{C}$ endowed with the weak topology. It is also endowed with a sheaf of analytic functions.

The basic example is the analytification of $\mathrm{Spec}(\mathbf{C})$, which may be explicitly described as

$$\mathrm{Spec}(\mathbf{C})^{\mathrm{hyb}} = \{|\cdot|_\infty^\varepsilon, 0 \leq \varepsilon \leq 1\},$$

where $|\cdot|_\infty^0 := |\cdot|_0$.

Let X be a complex algebraic variety. By functoriality, the structure morphism $\pi\colon X \to \mathrm{Spec}(\mathbf{C})$ gives rise to a morphism $\pi^{\mathrm{hyb}}\colon X^{\mathrm{hyb}} \to \mathrm{Spec}(\mathbf{C})^{\mathrm{hyb}}$ whose fibers we can describe: we have $(\pi^{\mathrm{hyb}})^{-1}(|\cdot|_0) = X_0^{\mathrm{an}}$ and, for each $\varepsilon \in (0,1]$, we have $(\pi^{\mathrm{hyb}})^{-1}(|\cdot|_\infty^\varepsilon) \simeq X^{\mathrm{h}}$.

To sum up, we obtain a locally ringed space with complex analytic fibers that seem to "degenerate" on a non-archimedean fiber. Such spaces have been used by V. Berkovich in [4] to give a topological interpretation (in an analytic space over $\mathbf{C}_0$) of the weight zero part of the limit mixed Hodge structure of a degenerating family of compact complex manifolds. They can also be found in the work [5] of S. Boucksom and M. Jonsson about the asymptotic behavior of volume forms in the same setting.

3. The compactification

Let X be a complex algebraic variety. We set

$$X^+ := X^{\mathrm{hyb}} - X^{\beth}.$$

Since $X^{\beth}$ is a closed subset of X_0^{an}, which is itself closed in X^{hyb}, X^+ is an open subset of X^{hyb}. In particular, it inherits a structure of locally ringed space. Denote by π^+ the restriction of π^{hyb} to X^+. We have

$$(\pi^+)^{-1}(|\cdot|_0) = X_\infty$$

and, for each $\varepsilon \in (0,1]$,

$$(\pi^+)^{-1}(|\cdot|_\infty^\varepsilon) = (\pi^{\mathrm{hyb}})^{-1}(|\cdot|_\infty^\varepsilon) \simeq X^{\mathrm{h}}.$$

The resulting space is not compact in general and contains several copies of X^{h}. To solve this issue, we will identify the points in the space X^+ that correspond to equivalent seminorms, *i.e.* seminorms that can be obtained one form the other by raising to some power $\lambda \in \mathbf{R}_{>0}$. Denote by $X^{\urcorner}$ the quotient space. We turn it into a locally ringed space by endowing it with the push-forward of the structure sheaf on X^+.

The archimedean part of the space $X^\urcorner$ now consists in exactly one copy of X^{h}. The non-archimedean part, which is the quotient of X_∞ by the equivalence of seminorms, is a so-called normalized space, as introduced by L. Fantini in [6].

Theorem 1. *The space $X^\urcorner$ is Hausdorff and compact and the map*

$$X^h = (\pi^+)^{-1}([|\cdot|_\infty]) \longrightarrow X^\urcorner$$

is an open embedding.

The map $X \mapsto X^\urcorner$ has additional properties. For instance, it is functorial with respect to proper morphisms.

Finally, to a coherent sheaf F on X, one may functorially associate a coherent sheaf $F^\urcorner$ on $X^\urcorner$. We have a GAGA theorem in this setting.

Theorem 2. *The functor*

$$F \in \mathrm{Coh}(X) \longmapsto F^\urcorner \in \mathrm{Coh}(X^\urcorner)$$

is an equivalence of categories.

For each coherent sheaf F on X and each $q \geq 0$, we have a natural isomorphism

$$H^q(X, F) \xrightarrow{\sim} H^q(X^\urcorner, F^\urcorner).$$

Note that the space $X^\urcorner$ has an open subset isomorphic to X^{h}. As a consequence, the space $X^\urcorner$ may be used to relate the categories of coherent sheaves over X and X^{h}.

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