AFFINOID SPACES OVER Z

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When developing analytic geometry over $\mathbf{Q}_p$, the first objects to consider are the so-called Tate algebras $\mathbf{Q}_p\{T_1, \dots, T_n\}$. They contain the power series with coefficients in $\mathbf{Q}_p$ that converge on the closed unit disk of center 0 in $\mathbf{Q}_p^n$. Remark that this last condition behaves well thanks to the non-archimedean triangle inequality.

In the archimedean setting, one needs to modifiy it and is led to consider instead "overconvergent Tate algebras" made of power series that converge in some arbitrary neighborhood of the closed unit disk. The same construction actually works over the ring of integers $\mathbf{Z}$. Generalizing slightly, for $r_1, \dots, r_n > 0$, we define $\mathbf{Z}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}^\dagger$ to be the ring of power series with coefficients in $\mathbf{Z}$ that converge in some neighborhood of the closed disk $\bar{D} = \bar{D}(0, (r_1, \dots, r_n)) \subset \mathbf{C}^n$.

To develop p-adic analytic geometry, one starts by studying the algebraic properties of those Tate algebras and, in particular, showing that they are noetherian. In order to do so, techniques that are quite specific to the non-archimedean setting are used, most notably the reduction map that enables to pass from a ring of power series over k to a ring of polynomials over the residue field $\tilde{k}$. Over $\mathbf{Z}$ (and over $\mathbf{C}$ too), such methods do not exist and the noetherianity result appears to be much more challenging. To the best of the knowledge of the author, until very recently, the only available result in this direction was the following theorem of D. Harbater, for $n = 1$.

Theorem 1 ([Har84, theorem 1.8]). *For every $r > 0$, the ring $\mathbf{Z}\{r^{-1}T\}^\dagger$ is noetherian.*

The proof is quite technical and relies on explicit descriptions. It is very unlikely that such a strategy can be made to work for a larger number of variables.

1. The complex setting

When replacing $\mathbf{Z}$ by $\mathbf{C}$, the analogous result is known, as a consequence of the following theorem of J. Frisch.

Theorem 2 ([Fri67, théorème I, 9]). *Let X be a complex analytic space and K be a compact subset of X that is semi-analytic and Stein. Then, the ring $\mathscr{O}(K)^\dagger$ of analytic functions that converge in some neighborhood of K is noetherian.*

Recall that a subset K of a complex analytic space $X$is said to be *semi-analytic* if it is locally defined by a finite number of inequations involving analytic

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functions and that it is said to be *Stein* if, for every coherent sheaf $\mathscr{F}$ defined in a neighborhood of K, we have

- for every $x \in K$, the stalk $\mathscr{F}_x$ is generated by the set of global sections $H^0(K, \mathscr{F})^\dagger$ (Cartan's theorem A);
- for every $q \geq 1$, $H^q(K, \mathscr{F}) = 0$ (Cartan's theorem B).

The proof is very geometric and makes a crucial use of the following properties:

(1) the local rings $\mathscr{O}_x$ are noetherian;
(2) the structure sheaf $\mathscr{O}$ is coherent;
(3) every compact semi-analytic space has finitely many connected components.

When applied to the compact $K = \bar{D}(0, (r_1, \ldots, r_n)) \subset \mathbf{C}^n$, the theorem shows that the ring $\mathbf{C}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}^\dagger$ is noetherian.

2. Berkovich analytic spaces over $\mathbf{Z}$

In order to use a strategy that is similar to the one used in the complex setting, one needs to have analytic spaces with good properties on which the rings $\mathbf{Z}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}^\dagger$ naturally appear as rings of functions on some compact sets. Berkovich analytic spaces over $\mathbf{Z}$ meet all those requirements.

Those spaces have been defined by V. Berkovich at the end of the first chapter of the monograph [Ber90]. Without going into the details, let us recall that the affine analytic space $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ of dimension n over $\mathbf{Z}$ is defined as the set of multiplicative semi-norms on $\mathbf{Z}[T_1, \ldots, T_n]$, endowed with the topology of pointwise convergence. An analytic function on this space is defined to be locally a uniform limit of rational functions without poles.

Since the absolute values over $\mathbf{Z}$ can be archimedean or not, the spaces $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ contain fibers that are non-archimedean (and look like p-adic Berkovich analytic spaces) and others that are archimedean (and look like complex analytic spaces, possibly modulo complex conjugation). Moreover, one may define a relative closed disk $\bar{\mathbf{D}} = \bar{\mathbf{D}}(0, (r_1, \ldots, r_n))$ around the 0 section in $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ and the ring of functions that converge in some neighborhood of this disk is exactly the ring $\mathbf{Z}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}^\dagger$ defined above. We refer to [Poi10b, annexe B] for a gentle introduction and to [Poi10a] for more details, including a complete study of the affine line.

3. Local properties

The article [Poi13] is devoted to the local study of analytic spaces over $\mathbf{Z}$. The main tool is a quite general local Weierstrass division theorem for the affine line over a Banach ring $(\mathscr{A}, \|\cdot\|)$ with mild conditions on $\mathscr{A}$ (that are automatically met if $\mathscr{A}$ is $\mathbf{Z}$ endowed with the usual absolute value or the completion of the ring of functions on a relative disk over $\mathbf{Z}$ for instance).

Theorem 3 ([Poi13, théorème 8.3])**.** *Denote by $\pi\colon X = \mathbf{A}^{1,\mathrm{an}}_{\mathscr{A}} \to B = \mathbf{A}^{0,\mathrm{an}}_{\mathscr{A}}$ the projection morphism. Let $b \in B$. Let $P \in \mathscr{H}(b)[T]$ be an irreducible polynomial and let x be the point of the fiber $X_b = \pi^{-1}(b)$ such that $P(x) = 0$. Let G be an element of $\mathscr{O}_{X,x}$ whose image in $\mathscr{O}_{X_b,x}$ is not zero.*

Then, there exists a non-negative integer m such that every element F of $\mathscr{O}_{X,x}$ may be written uniquely in the form $F = QG + R$, with $Q \in \mathscr{O}_{X,x}$ and $R \in \mathscr{O}_{B,b}[T]_{<m}$.

With this result at hand, one may deduce many local properties of the space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ by using a strategy that is close to the one used in the complex analytic setting.

Corollary 4.
- *For every point $x \in \mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$, the local ring $\mathscr{O}_x$ is henselian, noetherian, regular and excellent.*
- *The structure sheaf $\mathscr{O}$ on $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ is coherent.*

Thanks to the Weierstrass division theorem, one may also prove a sort of noetherianity result for coherent sheaves on $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$.

Corollary 5. *Let U be an open subset of $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$. Let $\mathscr{F}$ be a coherent sheaf on U and let $(\mathscr{F}_m)_{m\geq 0}$ be an increasing subsequence of subsheaves of $\mathscr{F}$. Then, every point x of U admits a neighborhood V in U such that the sequence $(\mathscr{F}_{m,|V})_{m\geq 0}$ is eventually constant.*

4. Global properties

In order to adapt the strategy of Frisch's proof, we also need to know that disk are Stein spaces. This is indeed the case.

Theorem 6. *Let $r_1, \dots, r_n > 0$. Set $\bar{\mathbf{D}} = \bar{\mathbf{D}}(0, (r_1, \dots, r_n)) \subset \mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$. Let $\mathscr{F}$ be a coherent sheaf defined in a neighborhood of $\bar{\mathbf{D}}$. Then, we have*
- *for every $x \in \bar{\mathbf{D}}$, the stalk $\mathscr{F}_x$ is generated by the set of global sections $H^0(\bar{\mathbf{D}}, \mathscr{F})^\dagger$ (theorem A);*
- *for every $q \geq 1$, $H^q(\bar{\mathbf{D}}, \mathscr{F}) = 0$ (theorem B).*

Let us explain some consequence of there results for affinoid spaces over $\mathbf{Z}$. Let us first give a definition in the spirit of the classical definition of affinoid spaces in rigid geometry. Consider a disk $\bar{\mathbf{D}} = \bar{\mathbf{D}}(0, (r_1, \dots, r_n)) \subset \mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ and a finite number of functions $f_1, \dots, f_m \in \mathscr{O}(\bar{\mathbf{D}})^\dagger$. Set

$$V = V(f_1, \dots, f_m) = \{x \in \bar{\mathbf{D}} \mid \forall i \in [\![1, m]\!], f_i(x) = 0\}$$

and denote by j_V the inclusion of V in $\bar{\mathbf{D}}$. Let $\mathscr{I}$ be the sheaf of ideals on $\bar{\mathbf{D}}$ generated by $(f_1, \dots, f_m)$. An *overconvergent affinoid space over* $\mathbf{Z}$ is defined to be a space isomorphic to $(V, j_V^{-1}(\mathscr{O}_{\bar{\mathbf{D}}}/\mathscr{I}))$. It is easy to deduce from theorem 6 that theorems A and B still hold for such spaces.

We would like to point out that those results are very similar to classical results in rigid analytic geometry: theorem A is analogous to Kiehl's theorem whereas theorem B resembles Tate's acyclicity theorem. For the former, this is clear. For the later, let us remark that a short argument involving the exact sequence $\mathscr{O}^m \xrightarrow{(f_1,\dots,f_m)} \mathscr{O} \to \mathscr{O}/\mathscr{I} \to 0$ and theorem B ensures that the global sections on V are exactly those one might expect:

$$\mathscr{O}(V) \simeq \mathbf{Z}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}^\dagger/(f_1, \dots, f_m).$$

This means that, if one would like to follow Tate's original construction and define a presheaf on an affinoid space by its global sections on its affinoid domains, then one would recover the structure sheaf we started with. In particular, this presheaf is a sheaf, which is one important part of Tate's acyclicity theorem.

5. Noetherianity

Let us finally go back to the noetherianity question we started with. The classical proof of Frisch's theorem uses a topological argument: compact semi-analytic subsets have only finitely many connected components. This is unknown in the theory of Berkovich analytic spaces over $\mathbf{Z}$, where the topological aspects are not well developed. (Let us however mention that T. Lemanissier recently proved that those spaces are locally arcwise connected in [Lem15]).

However, by using corollary 5 and the noetherianity of $\mathbf{C}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}^\dagger$, one is able to prove the expected result. Let us mention that this strategy is close to the one Langmann used in his proof of Frisch's theorem (see [Lan77]).

Theorem 7. *For every $r_1, \dots, r_n \in (0,1)$, the ring $\mathbf{Z}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}^\dagger$ is noetherian.*

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